On the predictability of economic structural change by the Poincaré-Bendixson theory

Denis Stijepic[*]

University of Hagen

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Abstract. The three-sector framework (relating to agriculture, manufacturing, and services) is one of the major concepts for studying the long-run change of the economic structure. We discuss the system-theoretical classification of the structural change phenomenon and, in particular, the predictability of the structural change in the three-sector framework by the Poincaré-Bendixson theory. To do so, we compare the assumptions of the Poincaré-Bendixson theory to (a) the typical axioms of structural change modelling, (b) the empirical evidence on the geometrical properties of structural change trajectories, and (c) some methodological arguments referring to the laws of structural change. The results of this comparison support the assumption that the structural change phenomenon is representable by a dynamic system that is predictable by the Poincaré-Bendixson theory. Moreover, we discuss briefly the implications of this result for structural change modelling and prediction as well as topics for further research.

Keywords. Poincaré-Bendixson theory, application, economics, structural change, labor reallocation, sectors, dynamics in the plane, simplex, trajectory, topology.
JEL codes. C61, C65, O41

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[*] Address: Lehrstuhl für Makroökonomik, Fernuniversität in Hagen, Universitätsstrasse 41, D-58084 Hagen, Germany. Phone: +4923319872640. Fax: +492331987391. Email: denis.stijepic@fernuni-hagen.de.

1. Introduction

The Poincaré-Bendixson theory, predicting the qualitative properties of the limit dynamics of smooth dynamic systems in the plane, is one of the fundaments of the dynamic systems theory. We discuss whether economic structural change is predictable by the Poincaré-Bendixson theory. While there are different notions of economic structure, we focus on the three-sector framework, where structural change is indicated by the long-run dynamics of labor reallocation across the sectors 'agriculture', 'manufacturing', and 'services'. The three-sector framework is one of the main concepts for studying the long-run dynamics of the macroeconomic structure. The related structural change and, in particular, industrialization and tertiarization are regarded as major drivers of economic development and have been analyzed in numerous studies over the last 200 years.[1]

The question whether the structural change in the three-sector framework can be represented by a dynamic system that is predictable by the Poincaré-Bendixson is the key to a *system-theoretical* characterization of the three-sector framework and its long-run dynamics.[2] Such a characterization is interesting for two reasons among others.

First, a 'proof' of the applicability of the Poincaré-Bendixson theory in the three-sector framework would not only allow us to characterize the *limit*-dynamics of structural change (i.e., to state that structural change is either transitory or cyclical in the limit, as we will see) but could also open the door for the application of other topological concepts, which can be used to predict the *transitional* structural change dynamics among others, in (a system-theoretical approach to) structural change analysis (see Stijepic (2015,2016,2017a) for an exploitation of the topological properties of structural change paths in structural change analysis). Such a characterization of transitional dynamics and limit-dynamics is usable for (long-run) structural change predictions, as demonstrated by Stijepic (2015,2017a).

Second, system-theoretical structural change models seem to be good complements to the standard structural change models (cf. Footnote 1). The latter generate relatively specific quantitative predictions of structural change. Yet, they are ideological to a great extent, since

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[1] For an overview of the structural change literature, see, e.g., Schettkat and Yocarini (2006), Krüger (2008), Silva and Teixeira (2008), Stijepic (2011, Chapter IV), Herrendorf et al. (2014), and Neuss (2018). Recent papers modelling structural change in the three-sector framework are, e.g., Kongsamut et al. (2001), Ngai and Pissarides (2007), Foellmi and Zweimüller (2008), Uy et al. (2013), and Stijepic (2015,2017a).

[2] To our knowledge, the Poincaré-Bendixson theory has not been applied in a system-theoretical study of the three-sector framework by now. Yet, there are applications of the Poincaré-Bendixson theory in other fields of economics (see Coles and Wright (1998) for an example).

they rely on very specific assumptions,[3] specific schools of thought (e.g., neoclassical, Keynesian, evolutionary, endogenous growth theory, etc.), and knife-edge parameter restrictions (cf. Stijepic (2011) and Temple (2003)), all of which are difficult to prove empirically and, thus, restrict the generality of models and the reliability of structural change predictions. In contrast, a system-theoretical modelling approach allows us to develop relatively general models generating relatively reliable predictions. Yet, such predictions are rather qualitative and encompass a relatively wide range of scenarios.

Our analysis of the predictability of structural change by the Poincaré-Bendixson theory can be classified as a system-theoretical contribution to structural change modelling. We do not seek to develop a fully specified economic model based on relatively specific and ideological assumptions regarding individual behavior and market-functioning but rather analyze whether the assumptions of the Poincaré-Bendixson theory are supported by (a) the mathematical implications of the standard structural change definition and other mathematical structural change modelling conventions, (b) the empirical evidence on the topological properties of structural change trajectories, and (c) the methodological arguments relating to the representation of economic laws by economic models. This analysis allows us to assess whether the dynamic system representing the laws of structural change is of the Poincaré-Bendixson type (and, thus, structural change can be predicted by the Poincaré-Bendixson theory).

Owing to the geometrical/topological nature of the Poincaré-Bendixson theory (and its proof) our analytical approach is rather geometrical and topological. In particular, we exploit the fact that the Poincaré-Bendixson theory is applicable to a trajectory family that *covers simply* a bounded and connected subset of the plane. Such a covering has straight-forward geometrical characteristics (among others, the trajectories are non-(self-)intersecting), which can be used to study the applicability of the Poincaré-Bendixson theory in structural change modelling, as we demonstrate.

Our results are in favor of the predictability of structural change by the Poincaré-Bendixson theory. Yet, some very interesting questions for further research, which are summarized in Section 9, remain open.

The rest of the paper is structured as follows. In Section 2, we formulate a meta-model of structural change representing the standard axioms and definitions of structural change

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[3] For example, Kongsamut et al. (2001) focus on a demand-side explanation of structural change, Ngai and Pissarides (2007) rely on supply-side structural change drivers, and Uy et al. (2013) choose an open-economy framework.

modelling. Section 3 recapitulates briefly the relevant (primarily topological) concepts for dynamic systems analysis. In Section 4, we summarize the empirical evidence regarding the topological properties of structural change paths. Section 5 discusses the methodological aspects of structural change modelling. Section 6 recapitulates the assumptions (and their geometrical interpretation) and the results of the Poincaré-Bendixson theory. Section 7 is devoted to the comparison of the Poincaré-Bendixson assumptions (Section 6) with the empirics (Section 4) and the axiomatics (Section 2) of structural change modelling by relying on the topological notions introduced in Section 3 and the methodological arguments derived in Section 5. In Section 8, we discuss the interpretation of the (results of the) Poincaré-Bendixson theory in the context of structural change and the structural change predictions resulting from this interpretation. A summary of the results and a discussion of the topics for further research is provided in Section 9.

2. A mathematical meta-model of structural change (axiomatics of structural change)

In this section, we recapitulate the mathematical meta-model of structural change that represents the modelling conventions of the structural change literature, as introduced by Stijepic (2015,2016,2017a). This model is fully specified by Definition 1 and Axioms 1-3. While Definition 1 relates model variables to the measures (in particular, employment shares) that are used to describe structural change in reality, Axioms 1-3 represent all the standard structural change modelling assumptions that cannot be proved by empirical evidence. Like all other empirical sciences, economics cannot work without relying on empirically improvable assumptions.

While there are different mathematical notational conventions, we choose the following notation for reasons of simplicity: small letters (e.g., x), bold small letters (e.g., $\mathbf{x}$), and capital letters (e.g., X) denote scalars, vectors/points, and sets, respectively. A dot indicates a derivative with respect to time (e.g., $\dot{x}$ is the derivative of x with respect to time).

Definition 1. *The sectors 1, 2, and 3 represent the primary (or agricultural) sector, the secondary (or manufacturing) sector, and the tertiary (or services sector), respectively. $y_{1c}(t)$, $y_{2c}(t)$, and $y_{3c}(t)$ denote the employment in sector 1, 2, and 3 at time $t \in D \subseteq R$ in country $c \in C$, respectively, where R is the set of real numbers. $y_c(t)$ is the aggregate employment at time t in country c. $x_{ic}(t) := y_{ic}(t)/y_c(t)$ is the employment share of sector i at time t in country c. The vector $\mathbf{x}_c(t) := (x_{1c}(t), x_{2c}(t), x_{3c}(t))$ indicates the cross-sector labor allocation at time t in*

country c. The term 'structural change (over the period [a,b]) in country c' refers to the long-run dynamics of the labor allocation $\mathbf{x}_c(t)$ (over the period [a,b]). The time point $t = 0$ represents the initial time point, and $\mathbf{x}_c(0) \equiv (x_{1c}(0), x_{2c}(0), x_{3c}(0))$ stands for the initial labor allocation in country c. If the initial labor allocation is given, we denote it by $\mathbf{x}_c^0 \equiv (x_{1c}^0, x_{2c}^0, x_{3c}^0)$.

Consider the three-dimensional real space (R^3) and the Cartesian coordinate system (x_1, x_2, x_3). We define the following set of points in it:

(1) $S := \{\mathbf{x} \equiv (x_1, x_2, x_3) \in R^3: x_1 + x_2 + x_3 = 1 \wedge \forall i \in \{1,2,3\}\ 0 \le x_i \le 1\}$

It is well known that (a) S is a two-dimensional standard simplex (henceforth, 2-simplex), (b) the 2-simplex is a triangle, and (c) the Cartesian coordinates of its vertices are the following:

(2) $(1,0,0) =: \mathbf{v}_1$ $(0,1,0) =: \mathbf{v}_2$ $(0,0,1) =: \mathbf{v}_3$

For an illustration, see Figure 1. Henceforth, we omit the coordinate axes, as depicted in the right panel of Figure 1.

Figure 1. *The 2-simplex in the Cartesian coordinate system (x_1, x_2, x_3) with and without coordinate axes.*
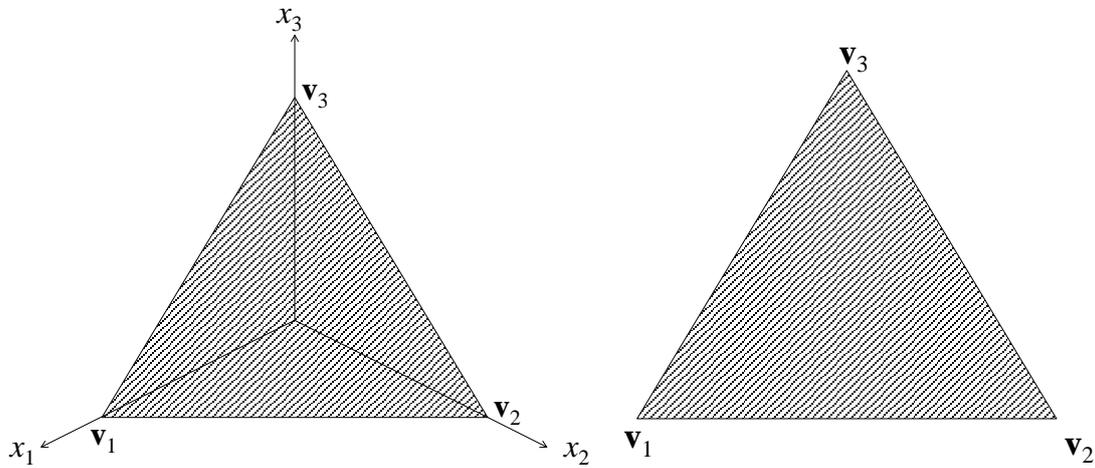
***Axiom 1.** (Cf. (1) and Definition 1.) The labor allocation function $\mathbf{x}_c(t)$: $D \times C \to S \subset R^3$ maps time $t \in D \supseteq [0,\infty)$ and the country index $c \in C$ to the 2-simplex S (which is located in the coordinate system (x_1, x_2, x_3)).*

In particular, for a given country index $c \in C$, the function $\mathbf{x}_c(t)$ assigns a point (x_1, x_2, x_3) on the 2-simplex S to each time point $t \in D$. Definition 1, Axiom 1, and (1) imply that the sectoral employment shares satisfy the following conditions:

(3) $\forall t \ \forall c \ \forall i \in \{1,2,3\} \ 0 \leq x_{ic}(t) \leq 1$,

(4) $\forall t \ \forall c \ x_{1c}(t)+x_{2c}(t)+x_{3c}(t)=1$,

and $\forall t \ \forall c \ y_c(t) = y_{1c}(t)+y_{2c}(t)+y_{3c}(t)$. These conditions are reasonable (e.g., the employment shares cannot be negative) and represent standard assumptions in structural change modelling (see Stijepic (2015,2016,2017a) for a discussion).

Axiom 2. $\forall t \in D \ \forall c \in C$, $\mathbf{x}_c(t)$ *is continuous in t.*

Axiom 3. $\forall t \in D \ \forall c \in C$, $\mathbf{x}_c(t)$ *is differentiable with respect to t.*

The assumption of continuous and differentiable functional forms is not only standard in structural change modelling but also in major parts of growth theory, development theory, and theories of long-run dynamics, in general. For example, the models presented by Kongsamut et al. (2001), Ngai and Pissarides (2007), Acemoglu and Guerrieri (2008), Foellmi and Zweimüller (2008), and Boppart (2014) are typical examples of multi-sector models satisfying Axioms 2 and 3 (cf. Stijepic (2015,2016,2017a)).

Among others, we use the model specified in this section to compare the empirical evidence on structural change to the assumptions of the Poincaré-Bendixson theory. To do so, we have to introduce some concepts for describing the dynamics of the model of this section. We turn now to this topic.

3. Topological concepts for describing structural change

In this section, we recapitulate some topological concepts for analyzing structural change, as discussed by Stijepic (2015,2016,2017a).

Axiom 1 and Definition 1 imply that we can study the structural change in country c by studying the properties of the function $\mathbf{x}_c(t)$. In our paper, we choose a topological approach for the study of this function, which relies on the concept of the image of the labor allocation function, as stated in Definition 2.

Definition 2. *The (image of the) structural trajectory (T_c) that depicts the structural change (cf. Definition 1) in country $c \in C$ over the time interval $G \subseteq D$ is defined as follows:*

(5) $T_c(G) := \{\mathbf{x}_c(t) \in S: t \in G\}$

Geometrically speaking, the economy c moves along $T_c(G) \subset S$ over the time period G. Thus, S is the *domain* of $T_c(G)$.

Trajectories can be characterized by using the (topological) concepts of continuity, self-intersection, and, in the case of two or more trajectories (where each trajectory represents the structural change in a different country), intersection. The intuitive/geometrical notion of continuity, self-intersection, and intersection is more or less obvious. For a *continuous* and *non-self-intersecting* trajectory, see Figure 2a; in contrast, Figures 2b and 2c depict examples of non-continuous and self-intersecting trajectories, respectively. Figure 2d depicts two intersecting trajectories, whereas Figure 2e depicts non-intersecting trajectories.

Figure 2. *Examples of (non-)continuous, (non-)self-intersecting, and (non-)intersecting trajectories on S.*
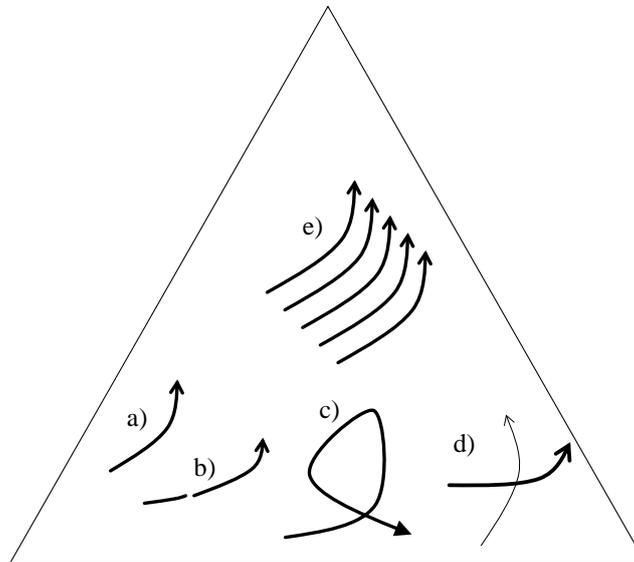
In our paper, we apply the following formal definitions of continuity, non-self-intersection, and non-intersection (cf. Stijepic (2015,2016,2017a)).

Definition 3. *The trajectory (5) is continuous on S (for a given $c \in C$) if the corresponding function $\mathbf{x}_c(t)$ is continuous (in t) on the interval G (for the given c).*

Definition 4. *The (continuous and non-closed) trajectory (5) is non-self-intersecting (for a given $c \in C$) if $\nexists(t_1,t_2,t_3) \in G^3$: $t_1 < t_2 < t_3 \wedge \mathbf{x}_c(t_1) = \mathbf{x}_c(t_3) \neq \mathbf{x}_c(t_2)$.*

Definition 5. *Two trajectories ($T_p(A)$ and $T_q(B)$, where $p,q \in C$ and $A,B \subseteq D$) intersect if $T_p(A) \cap T_q(B) \neq \emptyset$. Otherwise, if $T_p(A) \cap T_q(B) = \emptyset$, the trajectories $T_p(A)$ and $T_q(B)$ do not intersect.*

The Poincaré-Bendixson theory deals with the limit dynamics. The term *'limit dynamics'* refers to the dynamics for $t \to \infty$. The Poincaré-Bendixson theory relies on the concept of the 'omega limit set' for describing the limit dynamics.

Definition 6. *(Stijepic (2017a)) Let the function $\mathbf{x}_c(t)$ satisfy the Axioms 1-3. The point $\mathbf{x}_c^*$ is an omega limit point of the trajectory $T_c([0,\infty))$ (cf. (5)) if there exists a sequence of time points t_k (where $k=0,1,2,...$) that satisfies two conditions: (a) t_k converges to infinity (i.e., $t_k \to \infty$ for $k \to \infty$), and (b) the corresponding sequence $\mathbf{x}_c(t_k)$ converges to $\mathbf{x}_c^*$ (i.e., $\mathbf{x}_c(t_k) \to \mathbf{x}_c^*$ as $t_k \to \infty$). The omega limit set $O(T_c([0,\infty)))$ of the trajectory $T_c([0,\infty))$ is the union of all omega limit points of the trajectory $T_c([0,\infty))$.*
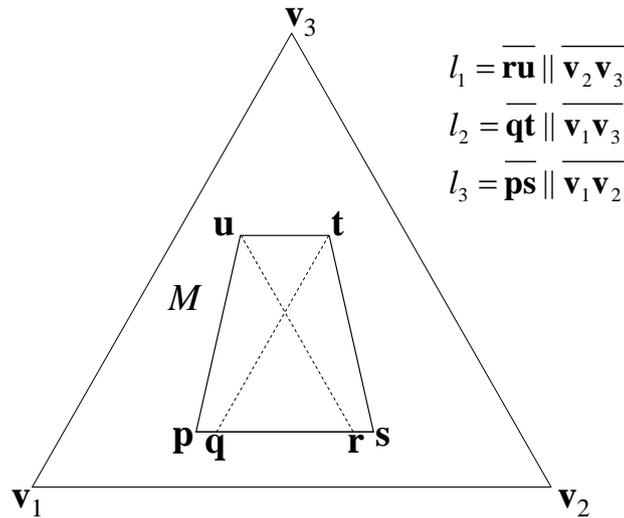
For a discussion and explanation of the omega limit set, see, e.g., Andronov et al. (1987, p.353ff.), Walter (1998, p.322), and Hale (2009, p.46f.). The (type of the) limit dynamics of an economy that moves along the trajectory image T_c is indicated by the omega limit set of the trajectory T_c, as we will see in Section 8. Finally, we introduce two (non-topological) concepts that help us to describe the empirical evidence discussed in Section 4.

Definition 7. *The set M is the minimal convex subset of S covering the trajectory family $B \subseteq C$ (cf. Definition 2) if (a) $M \subseteq S$, (b) M is convex, (c) $\bigcup_{c \in B} T_c([0,\infty)) \subseteq M$, and (d) among all the sets satisfying the criteria (a)-(c), M covers the smallest area of S.*

Definition 8. *Let M be a (convex) subset of S. Let L_1 be the set of all line-segments that (a) connect two boundary points of M and (b) are parallel to the line-segment $\overline{\mathbf{v}_2 \mathbf{v}_3}$. Let l_1 be the line-segment that has the greatest length among all the line-segments belonging to the set L_1 (cf. Figure 3). d_1 is the length of l_1. Analogously, let L_2 (L_3) be the set of all line-segments that (a) connect two boundary points of M and (b) are parallel to the line-segment $\overline{\mathbf{v}_1 \mathbf{v}_3}$ ($\overline{\mathbf{v}_1 \mathbf{v}_2}$).*

Let l_2 (l_3) be the line-segment that has the greatest length among all the line-segments belonging to the set L_2 (L_3). d_2 (d_3) is the length of l_2 (l_3).

Figure 3. *An example illustrating l_1, l_2, and l_3.*

$$l_1 = \overline{\mathbf{ru}} \parallel \overline{\mathbf{v_2 v_3}}$$
$$l_2 = \overline{\mathbf{qt}} \parallel \overline{\mathbf{v_1 v_3}}$$
$$l_3 = \overline{\mathbf{ps}} \parallel \overline{\mathbf{v_1 v_2}}$$

Definitions 7 and 8 allow us to measure the diameter of the minimal subset covering a family of trajectories in the three relevant directions on *S*. In fact, the diameter vector (d_1, d_2, d_3) is a (crude) measure of the heterogeneity of a trajectory family on *S*. The greater the diameters d_1-d_3, the greater the differences between the labor shares covered by the trajectories belonging to a trajectory family.

4. Empirical regularities of structural change (empirics of structural change)

While several stylized facts (or empirical regularities) of structural change are known (see Stijepic (2015,2016,2017a) for an overview), we focus on Regularities 1-3, which will prove useful for assessing the applicability of the Poincaré-Bendixson theory later. For empirical evidence on Regularities 1 and 3 and theoretical explanations, see Stijepic (2015,2016, 2017a). Figure 4 presents evidence supporting Regularities 1-3.

Regularity 1. *(Stijepic (2015,2016,2017a)) The long-run dynamics of labor allocation in the three-sector framework can be represented by non-self-intersecting trajectories.*

Regularity 2. *The labor-allocation trajectories differ significantly across countries. The minimal convex set M covering the trajectories representing the labor-allocation dynamics of*

the OECD countries over the last 200 years is relatively large (cf. Definition 7). In particular, the diameters d_1, d_2, and d_3 of the set M are relatively large (cf. Definition 8).

Regularity 3. *(Stijepic (2016)) The (long-run) labor allocation trajectories of different countries intersect.*

Figure 4. *Labor allocation trajectories of USA, France, Germany, Netherlands, UK, Japan, China, and Russia.*
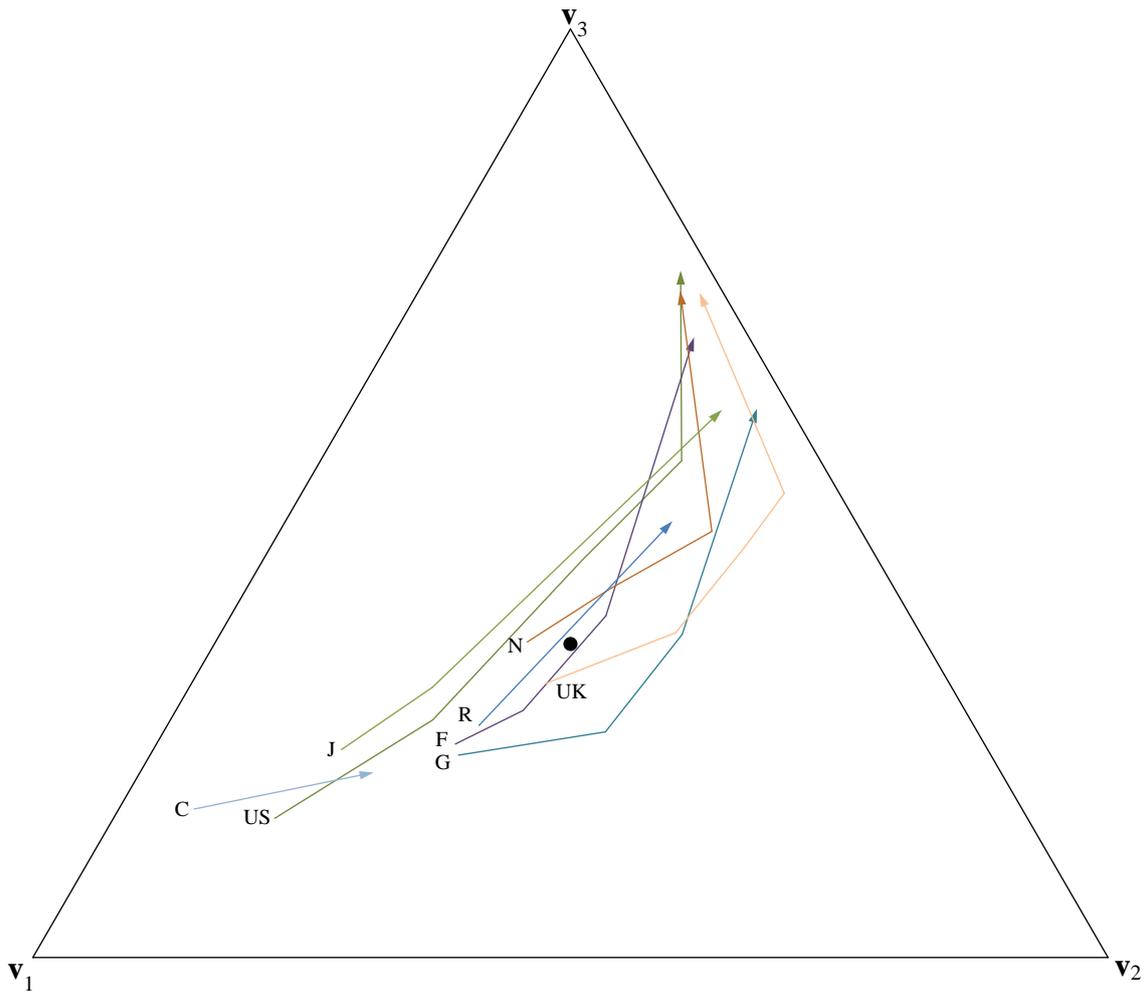
Notes: Data source: Maddison (1995). The black dot represents the barycenter of the simplex. Abbreviations: C – China, F – France, G – Germany, J – Japan, N – Netherlands, R – Russia, US – United States, UK – United Kingdom. Data points (years in parentheses): USA (1820, 1870, 1913, 1950, 1992), France (1870, 1913, 1950, 1992), Germany (1870, 1913, 1950, 1992), Netherlands (1870, 1913, 1950, 1992), UK (1820, 1870, 1913, 1950, 1992), Japan (1913, 1950, 1992), China (1950, 1992), Russia (1950, 1992).

5. Some methodological aspects

5.1 Topological properties of models representing laws

Economic modelling and predicting, as modelling and predicting in other sciences, relies on the assumption that the concepts/variables being modelled obey some sort of laws that can be expressed by mathematical equations/relations and are valid for different states of the environment and across time to some extent (since, otherwise, modelling and predicting would be obsolete). In particular, it is preferable that a (structural change) model (cf. Footnote 1) generates a family of curves (e.g., $\mathbf{x}(t,\mathbf{x}^0,\mathbf{p})$: $G{\times}S{\times}P{\rightarrow}S$; cf. Definition 1) that represent different initial states ($\mathbf{x}^0$) of the system and different parameter settings ($\mathbf{p}$) of the model. Given a connected state space (S), a connected parameter space (P), and an empirically observed 'typical' trajectory that is representable by a generic model trajectory ($\mathbf{x}(t,\mathbf{x}^0,\mathbf{p})$: $G{\rightarrow}S$, $\mathbf{x}^0{\in}S$, $\mathbf{p}{\in}P$), a model (representing laws) should be able to generate a new generic trajectory for a marginal deviation of the initial state from the observed initial state $\mathbf{x}^0{\in}S$ (and for a marginal deviation of model parameters from the assumed/observed parameter setting $\mathbf{p}{\in}P$; cf., e.g., Andronov et al. (1987, p.374 and p.405)). In other words, the trajectory family generated by a model representing (economic) laws should not only contain a generic trajectory that is sufficiently similar to an empirically observed 'typical' trajectory but should also cover completely a connected subset (H) of the state space (S) containing the generic model trajectory that represents the typical empirical trajectory.[4] Otherwise, the model would not be empirically relevant since initial states, parameters, and observed states (i.e., the typical trajectory) are not measurable exactly and we must assume that the 'real' initial states, parameters, and trajectories deviate from their observed/measured values and from the generic model trajectory representing them (cf. Andronov et al. (1987, p.374 and p.405)).[5] As we will see in Section 6.1, the extendibility of model predictions to a connected subset of the state (and parameter) space is a characteristic of (smooth) differential equation systems and, in particular, of the differential equation systems to which the Poincaré-Bendixson theory applies. For these reasons (among others) such systems are widespread in modelling of physical systems obeying physical laws.

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[4] The analogous is true for a parameter setting variation: the representative trajectory family should contain trajectories for all values of a connected subset (Q) of the parameter space (P) containing the parameter setting of the representative trajectory, to ensure 'coarseness'.

[5] For example, a model that can explain the dynamics of country c if the initial agricultural share in country c is equal to 0.78, but not if it is equal to 0.781, is a knife-edge model. Such knife-edge models are criticized in economics (see, e.g., Temple (2003)). Not to mention that we are not able to exactly measure the agricultural employment share in any country.

This discussion has a major implication for the meta-model presented in Section 2, as discussed in Section 5.2.

5.2 Implications for the meta-model of Section 2

Empirical evidence implies that structural change seems to follow some 'laws' (or rules/regularities), which are quite persistent across countries and time (see Stijepic (2017a) for a summary and literature references). *Economic* laws can be understood as ceteris paribus laws (cf., e.g., Jackson and Smith (2005) and Reutlinger et al. (2015)). This is reflected by the fact that standard structural change models (cf. Footnote 1) are dependent on parameters (e.g., time-preference rate and technology parameters) that, in general, may vary across countries. A most obvious example of observable cross-country parameter variation are the technology differences between developed and underdeveloped economies. Moreover, as implied by Regularity 2, the structural trajectories of the OECD countries (covering the last 200 years) differ significantly, such that we have to assume that the initial states $\mathbf{x}_c^0$ (cf. Definition 1) differ significantly across countries c (when choosing an initial time point within the last 200 years, which is a standard choice in structural change modelling). Thus, it makes sense to assume that the meta-model of Section 2 representing the dynamics of different countries (indexed by the index c and the index set C) represents different initial states and different parameter values of a structural change model. Then, the arguments elaborated in Section 5.1 imply that if the meta-model of Section 2 represents economic (structural change) laws and is of empirical relevance, we should assume that the trajectory family defined by Axioms 1-3 (and indexed by the set C) covers completely a connected subset H of the state space S, i.e.,

(6) $\bigcup_{c \in C} T_c([0,\infty)) \supseteq H,$

where the index set C, the relevant state space H, and the trajectories T_c represent at least some countries that have typical structural change patterns.

6. The Poincaré-Bendixson theory

By now, we have described the empirics and the axiomatics of structural change analysis by using the topological concepts introduced in Section 3. Now, we turn to the Poincaré-Bendixson theory. In particular, we describe the type of dynamic system to which the Poincaré-Bendixson theory applies (Poincaré-Bendixson assumptions), the geometrical characteristics that can be used to identify this type of system, and the limit-dynamics predictions of the Poincaré-Bendixson theory (Poincaré-Bendixson results).

6.1 The Poincaré-Bendixson assumptions and their topological interpretation

First, we recapitulate some elementary differential equation theory. Consider an autonomous differential equation system in the plane associated with the following initial value problem:

(7) $\dot{\mathbf{y}}(t) = \Phi(\mathbf{y}(t))$, $\mathbf{y}(0) = \mathbf{y}_0 \in H \subseteq R^2$, $\Phi: H \rightarrow R^2$, $t \in D \supseteq [0, \infty)$,

where Φ is a vector function, D an interval, and H an open and connected subset of R^2. A *solution* of the initial value problem (7) on the interval D is a vector function $\mathbf{y}(t): D \rightarrow H$, satisfying $\mathbf{y}(0) = \mathbf{y}_0$.

If the function Φ is sufficiently smooth, then there exist unique *solutions* of the initial value problem (7) for all $\mathbf{y}_0 \in H$ (see, e.g., Walter (1998), p.108 and p.110f.). For example, if Φ is analytical, *continuous* and continuously *differentiable*, or Lipschitz-continuous, then the uniqueness of solutions of (7) is ensured (see, e.g., Stijepic (2015, p.84f.) for a list of (exact) conditions ensuring uniqueness of solutions and literature references). The Poincaré-Bendixson theory applies to the differential equation systems in the plane (of the type (7)) that are sufficiently smooth to ensure the uniqueness of their solutions (see, e.g., Andronov et al. (1987, p.351ff.), Guckenheimer and Holmes (1990, p.43f.), Hale (2009, p.51ff.), and Teschl (2011, Chapter 7.3)). Henceforth, we name these systems 'sufficiently smooth autonomous differential equation systems in the plane' (abbreviated 'SSADES').

Given the set H of initial values, a SSADES generates a family (F) of trajectories/curves that have the following (topological) characteristics (cf. Walter (1998, p.10, p.36, and p.110f.), Hale (2009, p.18f. and p.38f.), and Stijepic (2015, p.84f.)).

Property 1. *(Continuity.) Each trajectory belonging to the family F is continuous (cf. Definition 3).*

Property 2. *(Non-self-intersection.) Each trajectory belonging to the family F is non-self-intersecting (cf. Definition 4).*

Property 3. *(Non-intersection.) The trajectories belonging to the family F are non-intersecting (cf. Definition 5).*

Property 4. *(Simple covering.) Let the (regular) point* $\mathbf{p} \in H$ *(cf. (7)) be located on the trajectory* τ, *which belongs to the family F. If we marginally deviate from the point* $\mathbf{p}$ *such*

that we reach the (regular) point **q**∈*H, then* **q** *is either on the trajectory τ or on another trajectory belonging to the trajectory family F (but not both).*

Finally, note that the Poincaré-Bendixson theory applies only to bounded trajectories (cf. Section 6.2). Thus, we postulate the following property.

Property 5. *(Boundedness.) The trajectory under consideration is bounded.*

Overall, Properties 1-5 characterize SSADES to which the Poincaré-Bendixson theory applies. Thus, we will use these properties to discuss/test whether the structural change axioms/empirics are consistent with the assumptions of the Poincaré-Bendixson theory. Note that even non-autonomous differential equation systems can generate trajectory families that have the Properties 1-5 and that are predictable by the Poincaré-Bendixson theory. Thus, Properties 1-5 constitute rather a test of the applicability of the Poincaré-Bendixson theory (in structural change modelling) than a test of the autonomy of the differential equation system (describing the dynamics of structural change).

Note that the Poincaré-Bendixson theory is restricted to the dynamic systems or the subsets (*H*) of the state space that do not generate/contain an infinite number of fixed points (see, e.g., Andronov et al. (1987, p.351f.) and Hale (2009, p.55)). Since we cannot evaluate economically or empirically this requirement/assumption, we assume, henceforth, that it is satisfied (i.e., the number of fixed points is finite). Giving an intuitive economic meaning to this assumption is an interesting topic for further research. Moreover, there are generalizations of the Poincaré-Bendixson theory (e.g., Solntzev (1945)) allowing for infinitely many fixed points (see, e.g., Ciesielski (2012, p.2117f.) and Nikolaev and Zhuzhoma (1999, p.37)). Further research could use these generalizations as a starting point.

6.2 The Poincaré-Bendixson results

Let $Y([0,\infty)):=\{\mathbf{y}(t)\in H: t\in[0,\infty)\}$ denote (the image of) a *bounded* (half-)trajectory on an open and connected subset (*H*) of the plane R^2, where the function $\mathbf{y}(t): D\rightarrow H$ represents the solution of a SSADES in the plane (for some given initial value $\mathbf{y}_0\in H$) and $D\supseteq[0,\infty)$ is an (open) interval (cf. Section 6.1).[6] The Poincaré-Bendixson theory states that then, one of the following statements is true (cf. Definition 6, Andronov et al. (1987, p.362f.), Guckenheimer

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[6] Recall that we assume here that the SSADES is such that it does not generate an infinite number of fixed points on the domain *H*, as explained at the end of Section 6.1.

and Holmes (1990, p.45), Hale (2009, p.55 and, in particular, Theorem 1.3) and Teschl (2011, Chapter 7.3 and, in particular, Theorem 7.16)):

(i) $O(Y([0,\infty)))$ is a fixed point (critical point).

(ii) $O(Y([0,\infty)))$ is (the image of) a Jordan curve.

(iii) $O(Y([0,\infty)))$ is (the image of) a homoclinic orbit (including its fixed point).

(iv) $O(Y([0,\infty)))$ is a union of at least two fixed points and the (images of the) trajectories connecting them ('heteroclinic union').

Note that (a) the term 'heteroclinic union' is not common in the literature and we use it here only as an abbreviation and (b) a 'heteroclinic union' must contain heteroclinic trajectories and can contain homoclinic trajectories.

We interpret and discuss the cases (i)-(iv) in the context of structural change in Section 8.

7. Comparison of the Poincaré-Bendixson assumptions to the axiomatics and empirics of structural change analysis via topology and methodology

Now, we can turn to the question of the applicability of the Poincaré-Bendixson theory in structural change modelling. Due to the spadework of the previous sections, this comparison is relatively straight-forward.

As discussed in Section 6.1, the Poincaré-Bendixson theory applies to a bounded trajectory generated by a dynamic system that generates a family of trajectories that are continuous, non-(self-)intersecting and simply cover a subset of the plane, i.e., the Poincaré-Bendixson theory applies to a family of trajectories characterized by the Properties 1-5. Our axiomatic model formulated in Section 2, the empirical evidence presented in Section 4, and the methodological arguments discussed in Section 5.2 are consistent with the assumption of such a system, as explained in the following.

Obviously, Axioms 1-3 support *Property 1*. *Property 2* is clearly supported by the empirical evidence (cf. Regularity 1).

Property 3 and Regularity 3 seem to contradict each other at first sight. However, Property 3 and Regularity 3 are not necessarily inconsistent. We have shown in Section 5.2 that (a) the structural change models represent ceteris paribus laws such that their predictions depend on parameters, which differ across countries, and (b) the countries differ by initial states $\mathbf{x}_c^0$ given the standard horizon of structural change analysis (of ca. 200 years). If the parameters of a SSADES (cf. Section 6.1) differ across countries and countries are characterized by different initial states $\mathbf{x}_c^0$, intersections of the trajectories representing different countries may arise (see the Appendix for an example). Thus, in general, Regularity 3 cannot be interpreted

as evidence against Property 3, and the study of the causes of the observable intersections (i.e., the analysis of the question whether Regularity 3 represents cross-country differences in parameters of the dynamic system generating the structural dynamics) is an interesting and very extensive topic for further research that could help to establish the validity of Property 3 (and the Poincaré-Bendixson theory) in structural change modelling. Moreover, there are theories/models (e.g., the Kongsamut et al. (2001) model and the Ngai and Pissarides (2007) model) that generate equilibriums[7] that represent the long-run economic dynamics and are characterized by non-intersecting labor allocation trajectories (see Stijepic (2016)); these models can be regarded as 'theoretical' arguments for modelling (long-run) structural change in a framework of non-intersecting trajectories (cf. Property 3).

The methodological arguments presented in Section 5.2 state that a structural change model should generate a family of trajectories that cover completely a bounded subset (*H*) of the state space. If we assume that a family (*F*) of trajectories satisfies this requirement and is characterized by Properties 1-3 (as discussed above), then obviously, this family (*F*) is characterized by *Property 4* as well. In particular, if a family of trajectories is continuous (cf. Property 1), non-self-intersecting (cf. Property 2), non-intersecting (cf. Property 3), and covers completely the domain *H* (according to the arguments of Section 5.2), then obviously, this family covers simply the domain *H* (cf. Property 4).

Finally, note that *Property 5* is consistent with the structural change meta-model presented in Section 2, since structural change is defined on the bounded domain *S* (cf. Axiom 1 and Figure 1).

8. On the interpretation/application of the Poincaré-Bendixson theory in the context of structural change

Now, assume that we impose additional restrictions (cf. Sections 4-7) on the structural change meta-model of Section 2 such that it becomes predictable by the Poincaré-Bendixson theory. According to Axiom 1, the structural change in the country $c \in C$ is given by the function $\mathbf{x}_c(t)$: $[0,\infty) \to S$, and the Poincaré-Bendixson theory predicts that the omega limit-set of this function is characterized by one and only one of the following scenarios (cf. Section 6.2 and Definition 2):

(i) $O(T_c([0,\infty)))$ is a fixed point (critical point).

(ii) $O(T_c([0,\infty)))$ is (the image of) a Jordan curve.

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[7] Kongsamut et al. (2001) (Ngai and Pissarides (2007)) name the long-run equilibrium arising in their model 'generalized balanced growth path' ('aggregate balanced growth path').

(iii) $O(T_c([0,\infty)))$ is (the image of) a homoclinic orbit (including its fixed point).

(iv) $O(T_c([0,\infty)))$ is a union of at least two fixed points and (the images of) the trajectories connecting them ('heteroclinic union').

We can interpret the scenarios (i)-(iv) as follows:

In case (i), the labor allocation in economy c converges along the trajectory image T_c to a fixed point ($\mathbf{x}_c^*$) for $t \to \infty$, i.e., $O(T_c([0,\infty)))=\{\mathbf{x}_c^*\}$. Thus, structural change is transitory, i.e., it comes to a halt in the limit.

Case (ii) is known from the Poincaré-Bendixson *theorem* (see, e.g., Miller and Michel (2007, p.290ff.) or Hale (2009, p.51ff.)). $O(T_c([0,\infty)))$ is the image of a Jordan curve in case (ii). Thus, T_c represents either (a) a closed trajectory (i.e., a Jordan curve) or (b) a non-closed trajectory converging to a closed trajectory (i.e., the labor allocation in economy c converges to a limit cycle).

In cases (iii) and (iv), for $t \to \infty$, the labor allocation in economy c converges along the trajectory image T_c to (all) the points of the homoclinic orbit or to (all) the points of the 'heteroclinic union', per definition of the term omega limit set (cf. Definition 6). Therefore, structural change is cyclical in the limit (cf. Figures 5 and 6, which are based on Andronov et al. (1987, p.362)).

Overall, the Poincaré-Bendixson theory allows us to reduce the number of potential structural change limit-scenarios to only two qualitative scenarios stating that structural change is either transitory or cyclical in the limit. The standard structural change literature (cf. Footnote 1) implies, in general, that structural change is transitory. Thus, our meta-model (i.e., the Poincaré-Bendixson theory) generalizes this result. The cyclicality of structural change is interesting since it implies, e.g., unemployment (associated with reallocation of labor across sectors) and fluctuations caused by structural change (e.g., productivity fluctuations). Moreover, cyclical structural change behavior seems to be 'inefficient' from the macroeconomic perspective since it causes structural change costs (see Stijepic (2016,2017b)). However, the empirical evidence shows that there are some short-run fluctuations with respect to structural change and, in particular, the structural change trajectories seem to have loops (on S) covering rather short periods of time (cf. Stijepic (2016,2017a)). Thus, cyclicality of structural change is not an irrelevant concept from the empirical point of view.

The representability of structural change by a system that is predictable by the Poincaré-Bendixson theory can have much more interesting implications if the statements on the geometrical properties of such a system (i.e., Properties 1-5) are combined with other laws or

empirical information on structural change. For example, Stijepic (2015) demonstrates by using an axiomatic/geometric approach that if the continuity and non-self-intersection properties (i.e., Properties 1 and 2) are combined with the fact that underdeveloped (advanced) economies are dominated by agriculture (services), the number of future structural change scenarios (in developed and emerging countries) can be reduced significantly, thus, yielding interesting meta-model predictions of structural change. Further research could focus on combining Properties 1-5 with empirical information on the geometrical properties of structural change trajectories and deriving predictions of structural change based on them (see Stijepic (2017a) for an overview of such empirically observable geometrical properties of structural change trajectories).

Figure 5. *Example: $O(T_c([0,\infty)))$ is the image of a homoclinic orbit.*
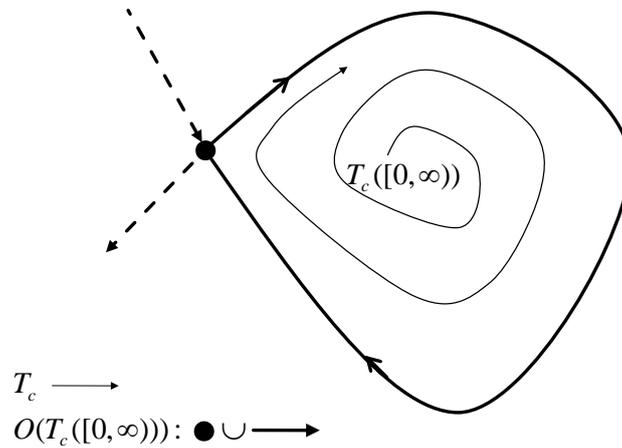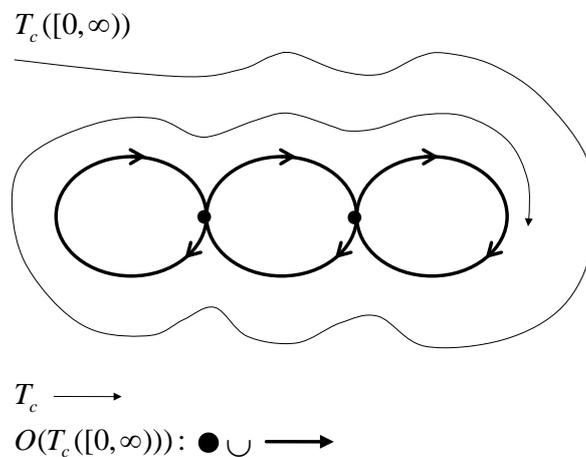

Figure 6. *Example: $O(T_c([0,\infty)))$ is a 'heteroclinic union'.*

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9. Concluding remarks

The Poincaré-Bendixson theory is a prototype system theory. It is applicable to a wide range of real world physical systems. While economic systems (which are often describable by ceteris paribus laws and non-autonomous differential equations) are quite different in comparison to physical systems, the long-run structural change in the three-sector framework seems to be quite similar to physical phenomena, since it obeys economic laws that are quite robust across countries and time. Therefore, we devoted this paper to the analysis of the question whether structural change behaves like one of the typical physical phenomena that can be described by the Poincaré-Bendixson theory.

We have chosen a rather innovative way of doing so. While it is possible to justify the applicability of the Poincaré-Bendixson theory in economics on the basis of theoretical assumptions (see, e.g., Coles and Wright (1998)), we have tried to rely on (a) the mathematical implications of standard structural change definitions and mathematical structural change modelling conventions, (b) empirical evidence, and (c) methodological arguments for establishing the validity of the Poincaré-Bendixson theory in structural change modelling aiming at a system-theoretical classification of the structural change phenomenon that is independent of specific economic assumption sets, schools of thought, and economic ideologies in general (cf. Section 1).

As noted in Section 7, it can be shown that some standard structural change models generate dynamic equilibriums that are consistent with (some of) the Poincaré-Bendixson assumptions (cf. Stijepic (2016)). However, neither does the application of the Poincaré-Bendixson theory in these models makes sense (since the assumptions of these models are sufficiently specific to solve the differential equation systems generated by them), nor generate these models the wide range of potential limit-scenarios (cf. Section 8) predicted by the Poincaré-Bendixson theory. Moreover, as discussed in Section 1, standard structural change models are characterized by many rather 'ideological' assumptions, such that their support of the applicability of the Poincaré-Bendixson theory in structural change modelling may be regarded as not very robust and, thus, our system-theoretical complement of these models seems to make sense.

The analysis of the applicability of the Poincaré-Bendixson theory in structural change modelling is in its core the search for an answer to the question whether structural change is representable by smooth autonomous differential equations (in the plane) or by simple coverings of a bounded and connected subset of a plane. The answer to this question is interesting, since (a) it allows us to classify one of the most persistent phenomena of

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economic development from the system-theoretical perspective and, thus, elaborate relatively robust (cf. Section 1) predictions of *limit*-structural change dynamics on the basis of the Poincaré-Bendixson theory and (b) the fact that structural change is representable by simple coverings implies in conjunction with other (empirical) structural change facts some predictions/statements regarding the nature of *transitional* structural change dynamics, as demonstrated by Stijepic (2015) (cf. Stijepic (2017a)). Further research could deal with the continuation of the latter approach for elaborating further aspects of *transitional* structural change dynamics.

Moreover, we have elaborated two relatively interesting topics for further research: (1.) the study of the case with infinitely many fixed points mentioned in Section 6.1 and (2.) the interpretation of the empirical evidence on the intersections of countries' structural change trajectories and, in particular, the question whether these intersections can be explained by cross-country parameter variation (cf. Section 7).

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Appendix

As discussed in Section 6.1, SSADES are characterized by non-(self-)intersecting trajectory families. In this appendix, we show that such a non-(self-)intersecting trajectory family can give rise to intersections of trajectories representing different countries if the parameter setting differs across countries.

Let (u,v) be a coordinate system that is parallel to S, such that all points on S can be identified by their coordinates (u,v). Assume, moreover, a family of lines representing the images of linear non-(self-)intersecting trajectories covering (a subset of) S and indicating a movement away from vertex $\mathbf{v}_1$. Let this family of lines be given by the family of functions $u = f_b(a,v) = a + bv$, where $f_b(a,v)$: $R^2 \rightarrow R$ for a given $b \in R$. Assume that country 1 is characterized by $b = 1/3$ and country 2 is characterized by $b = 2/3$. Figure A1 illustrates the family of functions $f_b(a,v)$, where the parameter setting $b = 1/3$ ($b = 2/3$) is represented by solid lines/arrows (dashed lines/arrows) and only a subset of the line-segments that intersect with S is depicted. Moreover, assume that the initial states $\mathbf{x}_c^0$ of the countries are not identical (i.e., $\mathbf{x}_1^0 \neq \mathbf{x}_2^0$) and, thus, at $t = 0$, the countries are not located at the same place on S. As we can see in Figure A1, the images of the trajectories of countries 1 and 2 intersect (at the point $\mathbf{z}$), although each of the countries is modelled by a system of non-(self-)intersecting trajectories and the countries differ only by the parameter b (and by the initial states $\mathbf{x}_c^0$).

Figure A1. *An example of trajectory intersection arising from cross-country differences regarding parameters and initial states.*