# Electoral competition with endogenous leadership

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**Abstract**

We study electoral competition using a new continuous-time framework for games with a second-mover advantage. Two candidates with different levels of valence and competence in implementing their announced policy, compete for office. There is often a unique (easily calculated) SPNE in which the stronger candidate leads, distorting their policy towards the status quo. In static analogs to this continuous time game, the unique equilibrium is usually in mixed strategies. In our continuous time setting, players never move simultaneously if this involves randomizing over their actions. In this sense, mixed strategy equilibria are an artifact of static games.

*Keywords*: timing, continuous-time game, second-mover advantage, electoral competition, leadership

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# 1 Introduction

Continuous time provides a natural setting for timing games. However, due to technical difficulties related to the non-existence of a first subgame after time $t$ in continuous time (Simon and Stinchcombe, 1989), many authors instead use static or discrete-time settings and impose an order of moves. This produces misleading results if the imposed order differs from the equilibrium order in a more general game where players are free to decide about their actions and their timing. Our electoral competition game demonstrates the power of continuous time modeling.

We begin with a static electoral competition game that illustrates candidates' second-mover advantage. Apart from special cases, the only simultaneous move Nash equilibrium is in mixed strategies (Section 2). The payoffs from this game are the input to the continuous time game where the order of candidates' policy announcements is endogenous. Our static game is novel because it includes candidates' competence, defined as the probability that they will implement their announced policy rather than maintain the status quo. This feature causes the status quo to affect equilibrium platforms, conditional on the order of moves.

The irrelevance of the status quo for candidates' strategic positioning in policy space limits most models of electoral competition. Persson and Tabellini (2000, p.151) write: *"A more bothersome feature ... is the unimportance of the status quo. In these models of electoral competition, and in particular in the median-voter model, history plays no role."* Introducing "competence" as a candidate characteristic captures the importance of the status quo policy. Voters evaluate candidates not only by their *announced* policies, but also consider the likelihood that the candidate will deliver on their promises. Voters consider the expectation and variance of the policy outcome conditional on a candidate winning the election.

As in many previous models, we also allow candidates to differ in their "valence", their innate appeal (e.g., charisma), independent of their policy position. Both valence and competence contribute to a candidate's electoral strength, but they have subtly different equilibrium effects.

We use a stochastic median-voter model. It is as if there were a single voter whose ideal policy is unknown to the candidates when they choose their policy platforms. The voter knows their ideal. Candidates are purely office-motivated, each candidate seeking to maximize the probability of their election.

We assume that one candidate is more competent and has higher valance, and thus is unambiguously stronger. If candidates must move simultaneously, there usually exists no pure strategy Nash equilibrium. Therefore, most static models of electoral competition with valence study mixed strategy equilibria (Aragones and Palfrey, 2002; Groseclose, 2001; Hummel, 2010). As with other anti-coordination games (e.g., Matching Pennies), the dominant candidate wants to match the rival's policy platform, because this assures winning the election; the weaker candidate seeks to differentiate its platform from the stronger candidate's in order to win under extreme realizations of the median voter's bliss point.

Candidates' incentives to lead or to follow are crucial for our dynamic analysis, so in the static setting we also consider the equilibrium outcome under exogenously chosen order of moves. Both candidates have a second-mover advantage, which is greater for the weaker candidate. Apart from special cases, the weaker candidate loses the election with certainty as the leader, but wins with positive probability as the follower. The stronger candidate, in contrast, wins with positive probability both as the leader and as the follower. If the stronger candidate leads, it chooses a policy platform near the center of the policy space, but distorted towards the status quo, unless the candidates are equally competent.[1]

Results differ slightly if both candidates have the same valence but one candidate is more competent. Here, if the weaker candidate leads and adopts the status quo policy, the other candidate loses its competence advantage: by matching the leader's platform, both candidates maintain the status quo policy after the election, so competence does not matter. In this case, the weaker candidate can win the election with positive probability even when leading.

Section 3 develops a general continuous time framework for two-player games with a second-mover advantage. This framework has many potential applications, e.g., in price competition games. We use the framework to endogenize candidates' timing (and order) of moves in the electoral competition setting (Section 4). Each of the two candidates can move at most once, choosing an action from an infinite action set, or can choose to refrain from making any move.

Our main result (Theorem 1) uses a small set of additional conditions to identify a unique subgame perfect Nash equilibrium (SPNE) in the continuous time game; here, players move sequentially. We exclude equilibria with simultaneous

[1] As leader, the stronger candidate chooses the policy that would maximize the rival's appeal to the expected median voter. Because the weaker follower must distinguish itself from the leader, the leader's action induces the follower to move away from a position that would otherwise be advantageous to it.

moves by assuming that in every subgame at least one player has an incentive to wait, if the other player were to behave as in a (hypothetical) equilibrium where candidates are *required* to move simultaneously at $t$. Proposition 3 provides the most important justification for this assumption. This proposition deals with the case where the only simultaneous move equilibrium is in mixed strategies, as in our electoral competition game. If (in the continuous time setting) player $i$ were to move at $t$, mixing over its actions, the rival $j$ strictly prefers to condition its action on the outcome of $i$'s randomization, over moving simultaneously. Player $j$ incurs no cost from the (negligible) delay that this conditioning requires.

Proposition 3 shows that imposing simultaneity if in reality players can choose when to move, might be restrictive and lead to erroneous results. Games without a pure strategy Nash equilibrium provide a leading example. Whereas in a one-shot game players then randomize in equilibrium, in continuous time a sequential pure strategy equilibrium typically replaces the mixed strategy equilibrium. Analyses of one-shot games should therefore justify the assumption that players must choose their actions simultaneously, e.g., because they have to move in a particular time frame, during which they cannot observe the rival's behavior.[2]

The results from the continuous time game may be not only more plausible, but also simpler to characterize than the mixed strategy equilibrium to the static game. Section 4 illustrates this. Here we use the payoffs from our static electoral competition game (from Section 2) to construct a continuous time game in which candidates choose the timing of their policy announcement, or refrain from running for office altogether. We add a simple time-dependency to candidates' payoffs, providing a number of reasons that a candidate does not want to announce a policy platform too late (or too early).

We then apply Theorem 1 to identify a unique sequential SPNE valid under a large set of parameter values. Typically, the stronger candidate is the leader and the weaker candidate enjoys the second-mover advantage. The logic underlying Theorem 1 requires working backwards in time. Weakness confers patience in this setting; the weaker candidate is more willing to out-wait their rival. Anticipating

[2]With penalty kicks the goalie typically cannot "wait an instant" to condition their response on the kicker's action, so here a mixed strategy equilibrium is sensible. Some types of poker also involve mixed strategies, even though players move sequentially; here the fact that players interact many times explains the use of mixed strategies. Mixed strategies are plausible in many circumstances. Our point is that they sometimes arise because a static model replaces an apparently more complicated but realistic dynamic game.

this, the stronger candidate then leads early in the game to avoid the cost of waiting. A candidate's innate advantage from greater valence and competence always leads to it having a larger equilibrium probability of election, despite the fact that this greater innate strength causes it to forgo the second-mover advantage in the continuous time setting.

To recap: For many games, including electoral competition, where there is no intrinsic reason that candidates must move simultaneously, a dynamic model is more descriptive than a static model. The dynamic game appears more complicated than its static analog. However, with the help of Theorem 1, the analysis of the dynamic game may be considerably simpler than for the static analog. Mixed strategy equilibria over continuous action sets are difficult to compute, making comparative statics of the static game equilibrium difficult to obtain.[3] In contrast, the sequential SPNE in our dynamic game is trivial to compute and is insensitive to precise functional form assumptions or parameter values. It merely requires checking a small set of general conditions needed to apply Theorem 1.

## Related literature

Our general framework presented in Section 3 builds on Simon and Stinchcombe (1989), who provide a rigorous analysis of continuous time games. Like them, we focus on pure strategies for players' timing decisions (and argue in Footnote 22 why this is not restrictive). We provide conditions under which agents also do not mix over their action choices. We simplify their setup by restricting the number of players to two, and allowing each player to move at most once.

The continuous time framework has an important advantage over discrete time, where if player 1 uses a pure strategy to determine their action choice in a period, an impatient player 2 strictly prefers to implement their best response in the same period. Therefore, in a discrete time setting with impatient players and perfect and complete information, no pure strategy equilibria with sequential moves exist, as there is always a profitable deviation. By contrast, in continuous time, the follower can move immediately after the leader without incurring a cost of waiting (Simon and Stinchcombe, 1989). A continuous time game thus opens the possibility of both simultaneous and sequential move equilibria in pure strategies.

For this and for other reasons, continuous time games are increasingly popular

[3] A common approach involves discretizing the action space. See Martin and Sandholm (2022).

among scholars. Examples include Bergin (1992), Bergin and MacLeod (1993), or Park and Xiong (2024) who use settings with inertia. Calford and Oprea (2017) analyze the effects of inertia using laboratory experiments and find that "behavior tends towards discrete time benchmarks as inertia grows large and perfectly continuous time benchmarks as it falls towards zero". We follow Simon and Stinchcombe (1989) and abstract away from exogenous lags in players' responses.[4]

Continuous time games are sometimes used to endogenize the order of players' moves. For example, Hoppe and Lehmann-Grube (2005) study an innovation timing game. Hendricks, Weiss, and Wilson (1988) analyze a War of Attrition. These papers focus on pure timing games, where a player's only decision is when to move. By contrast, we analyze players' non-trivial action choices in conjunction with their timing decisions. Unlike Simon and Stinchcombe (1989), we do not restrict action sets to be finite; here we rely on our companion paper, Karp et al. (2024), that – similar to Simon and Stinchcombe (1989) – formally establishes the relation between games in continuous time, and discrete time analogs with an infinitely fine grid.[5] Allowing for infinite action sets, such as intervals, vastly expands the set of potential applications, as our electoral competition game highlights.[6]

Our static electoral competition game is in the tradition of Downs (1957). One candidate has characteristics contributing to electoral strength. Aragones and Palfrey (2002) and Groseclose (2001) study electoral competition in models where candidates differ in valence but not in competence. As in our setting, (typically) the only equilibrium when imposing simultaneous moves is in mixed strategies. However, we use this static game to construct the payoffs of a dynamic game, where the unique SPNE is sequential, involving pure strategies. We show how candidates' characteristics determine the equilibrium order of their moves, as well as their policy platforms.[7]

[4]See also Sannikov (2007) for a continuous time setting that uses stochastic methods.

[5]Ambrus and Lu (2015) introduce a continuous-time framework for coalitional bargaining and show that Markov perfect equilibria are the only SPNE of their model that can be approximated by SPNE of nearby discrete-time bargaining models.

[6]Hamilton and Slutsky (1990) endogenize the order of players' moves in a two-period setting. As noted above, this discrete time setup cannot accommodate players' impatience. Therefore, these authors abstract away from discounting and all other forms of impatience. Other authors have used their framework to endogenize the order of moves in applications, using risk dominance considerations (Harsanyi and Selten, 1988). Examples include van Damme and Hurkens (1999 and 2004) who study quantity and price competition games.

[7]Battaglini (2014) studies electoral competition games with repeated elections, whereas we

Only a few papers have studied the effects of a difference in candidates' competence in the electoral competition setting. Miller (2011) comes closest to our approach; his "effectiveness" corresponds to our "competence", and he also includes valence. However, Miller (2011) considers the extreme case of purely policy-motivated candidates with strong biases: one candidate is interested only in seeing an implemented policy that is as far left as possible, and the other one as far right as possible. Abstracting away from uncertainty about the median voter's bliss point, for most parameter constellations, the stronger candidate wins the election. The weaker candidate's sole interest is to moderate the stronger candidate's policy choice. Consequently, the weaker candidate positions itself at the median voter's bliss point whereas the stronger candidate chooses a position as far away from the median voter's bliss point as possible while still winning the election. By contrast, we adopt the more standard assumption that candidates are office-motivated.

Desai and Tyson (2023) combine valence and competence into one characteristic called "capability". Capability affects the probability that a winning candidate implements their policy platform, similar to "competence" in our model; but voters also attach intrinsic value to competence, much as they do to valence. As in Miller (2011), the candidates in Desai and Tyson (2023) are purely policy-motivated, with one candidate having an ideal point to the left of the center and the other candidate to the right. They find that in equilibrium both candidates choose an expected policy more moderate than their ideal point, with the weaker candidate moving closer to the center than the stronger candidate.

Gouret and Rossignol (2019) model competence as a multiplier on the voter's (dis)utility for a candidate. As with our approach, voters prefer a candidate with low competence when its policy platform is far from the voter's ideal point. However, unlike our approach, Gouret and Rossignol's (2019) model does not contain a status quo; competence in their model is instead the intensity with which the proposed platform will be implemented.

The remainder of this paper is organized as follows. Section 2 introduces our electoral competition game, and analyzes it for different timing regimes: simultaneous vs. sequential with an exogenous order of moves. In Section 3, we introduce our general continuous time framework that can be used to analyze various ap-

analyze a single election. Kamada and Sugaya (2020) endogenize policy platforms and their timing in electoral competition games where candidates' opportunities to modify their positions arrive stochastically.

plications, including our electoral competition game. We focus on games with a second-mover advantage. Subsequently, in Section 4, we transform our electoral competition setting to a continuous time game, and use our main result from Section 3 (Theorem 1) to endogenize candidates' choices of policy platforms along with their timing. Section 5 concludes. Proofs are relegated to the Appendix. (The Online Appendix considers additional cases and contains a supplementary discussion of our dynamic electoral competition game.)

# 2 Electoral competition game

The median voter's ideal point is a random variable $l \sim U[-1,1]$.[8] Candidate $i \in \{A,B\}$ announces policy $x_i \in \mathbb{R}$ before learning the realization of $l$. If $i$ wins they implement their announced policy with probability $p_i > 0$, and retain the default policy $-1 < s < 1$ with probability $1-p_i$; $p_i$ measures the candidate's competence. If $i$ wins, voters obtain a utility premium $v_i$, unrelated to the policy; $v_i$ is $i$'s valence.

The median voter, who knows their ideal point, obtains the expected utility

$$u_l(i,x_i) = -p_i(l-x_i)^2 - (1-p_i)(l-s)^2 + v_i,$$

from choosing candidate $i$. The candidate who gets this agent's vote wins the election. The voter chooses candidate $i$ if $u_l(i,x_i) > u_l(j,x_j)$, $j \neq i$. In case of a tie, the voter chooses candidate $A$ with exogenous probability $\alpha \geq 1/2$.[9]

Candidates are purely office-motivated, so candidate $i$'s payoff equals their probability of winning the election, denoted $\pi_i$. This model is tractable in the dynamic environment where candidates decide when to announce their policy platforms. First, however, we analyze the game under three exogenous timing regimes: either $A$ or $B$ leads or they choose their platforms simultaneously. In the latter case, the game often does not have a pure strategy equilibrium.

We begin with the median voter's decision problem. If this voter chooses

[8] Duggan (2005) called this the stochastic preference model. Ashworth and Bueno de Mesquita (2009) include uncertainty about both the median voter's ideal point and the candidates' valence.

[9] The assumption $\alpha \geq 1/2$ is reasonable given that $A$ is the stronger candidate, and it reduces the number of case distinctions in the proof of Proposition 2 (below). Results remain qualitatively unchanged if the assumption is relaxed, as shown in Online Appendix B.1.

candidate $i$, the expected policy is

$$a_i \equiv p_i x_i + (1 - p_i)s \Leftrightarrow x_i = \frac{a_i - (1 - p_i)s}{p_i}. \tag{1}$$

If $a_A > a_B$, the median voter views candidate $A$ as the rightist choice. To simplify the algebra, we use $a_i$ instead of $x_i$ as the strategic variable[10]. We refer to $a_i$ as both candidate $i$'s *action* and as their expected policy (conditional on winning), whereas $x_i$ is $i$'s *policy platform*. With this transformation, the utility of the median voter who chooses candidate $i$ is:

$$U_l(i, a_i) = -2(l - s)(s - a_i) - \frac{(s - a_i)^2}{p_i} - (l - s)^2 + v_i. \tag{2}$$

We adopt the following notation: $\Delta a \equiv a_A - a_B$, $\Delta v \equiv v_A - v_B$, $\Delta p \equiv p_A - p_B$, and $\Delta U_l \equiv U_l(A, a_A) - U_l(B, a_B)$. Additionally, let

$$q_i \equiv -\frac{(s - a_i)^2}{p_i}$$

be the "competence-weighted effect" of candidate $i$'s expected policy distance from the status quo $s$. We also define $\Delta q \equiv q_A - q_B$. For $a = a_A = a_B$ we have:

$$\Delta q|_{\Delta a = 0} = \frac{\Delta p}{p_A p_B}(s - a)^2. \tag{3}$$

Our definitions imply

$$\Delta U_l = v_A - v_B - \frac{(s - a_A)^2}{p_A} + \frac{(s - a_B)^2}{p_B} + 2(l - s)(a_A - a_B). \tag{4}$$

This utility difference can be simplified to

$$\Delta U_l = \Delta v + \Delta q + 2(l - s)\Delta a. \tag{5}$$

The preference of the voter with ideal point $l$ between the two candidates depends on the candidates' valences, competence effects, and location effects. Higher valence is always an advantage to a candidate. The effect of greater competence depends on the relation between the candidate's position and the status quo. In the special case where the candidates announce the same expected policy ($a_A = a_B$),

[10]Note that by $x_i \in \mathbb{R}$, a candidate can choose any $a_i \in \mathbb{R}$ and specifically any $a_i \in [-1, 1]$.

Equation (3) implies that greater competence makes a candidate more attractive.[11]

Provided that $\Delta a \neq 0$, from Equation (5), we obtain the critical point $\bar{l}$ at which the voter is indifferent between the two candidates:

$$\bar{l}(a_A, a_B) = s - \frac{\Delta v + \Delta q}{2\Delta a} \tag{6}$$

$$= s - \frac{p_A p_B(v_A - v_B) - p_B(s - a_A)^2 + p_A(s - a_B)^2}{2p_A p_B(a_A - a_B)}. \tag{7}$$

Suppose that the indifferent median voter is "interior": $-1 < \bar{l}(a_A, a_B) < 1$.[12] If $\Delta a > 0$, then $A$ is the rightist candidate and wins with probability $\pi_A(a_A, a_B) = \frac{1}{2}(1 - \bar{l}(a_A, a_B))$. If instead, $A$ is the leftist candidate ($\Delta a < 0$), then $A$'s winning probability is $\pi_A(a_A, a_B) = \frac{1}{2}(1 + \bar{l}(a_A, a_B))$.[13]

If $\Delta a = 0$, Equation (5) simplifies to

$$\Delta U_l|_{\Delta a=0} = \Delta v + \Delta q|_{\Delta a=0}. \tag{8}$$

If one candidate dominates in both valence and competence (i.e., $\Delta v > 0$ and $\Delta p > 0$) and in addition $\Delta a = 0$, the voter prefers the dominant candidate for all $l$. If each candidate dominates in a different characteristic (and $\Delta a = 0$), then the voter prefers candidate $A$ if $\Delta v > -\Delta q|_{\Delta a=0}$.

We emphasize the case where candidate $A$ is dominant in both competence and valence. There we obtain a clear ranking of candidates' incentives to lead or follow, facilitating our later dynamic analysis. We also consider the special cases where candidates differ only in their valence ($\Delta p = 0$) or only in their competence ($\Delta v = 0$). Online Appendix B.2 examines the mixed case where each candidate dominates in one of these characteristics.[14]

[11] With $a_A = a_B \neq s$ the expected policy is the same under both candidates, but the policy platform of the more competent candidate is necessarily closer to the status quo. Therefore, the variance of the policy is smaller when the more competent candidate wins. Thus, the preference for the more competent candidate reflects the voter's implicit risk aversion.

[12] If the solution to Equation (6) does not satisfy $-1 \leq \bar{l} \leq 1$, then it is understood that $\bar{l}$ takes a boundary value. The proof of Proposition 1 uses this convention.

[13] We use the abbreviated form $\pi_A$ when there is no risk of confusion.

[14] This case can also be analyzed in continuous time, but requires complex case distinctions due to the conflicting effects of competence and valence.

## 2.1 Case: $\Delta v > 0$ and $\Delta p > 0$

With $\Delta v > 0$ and $\Delta p > 0$, $A$ is more competent and has higher valence, and therefore is the dominant candidate. To exclude the extreme case where $A$ wins with certainty both as the follower and as the leader, we assume

$$\Delta v < p_B \left(1 - \frac{\Delta p}{p_A} s^2\right). \tag{9}$$

Figure 1 helps to visualize the difference between the candidates' incentives, holding fixed $\Delta v, p_A, p_B, s$. In both panels, the black curves show the graphs of the critical voter's position, $\bar{l}(a_A, a_B)$. The left panel shows these graphs for fixed $a_B = 0.25$ as $a_A$ varies, and the right panel shows the graphs for fixed $a_A = 0.25$ as $a_B$ varies. Given an action pair, one-half the height of a region (green for $A$ and red for $B$) equals the probability that the agent wins. The figure shows that candidate $A$ has an incentive to move close to its rival, whereas $B$ does better by moving away. This difference is key to understanding why $B$'s payoff (its election probability) is zero if it leads, and positive if it follows.

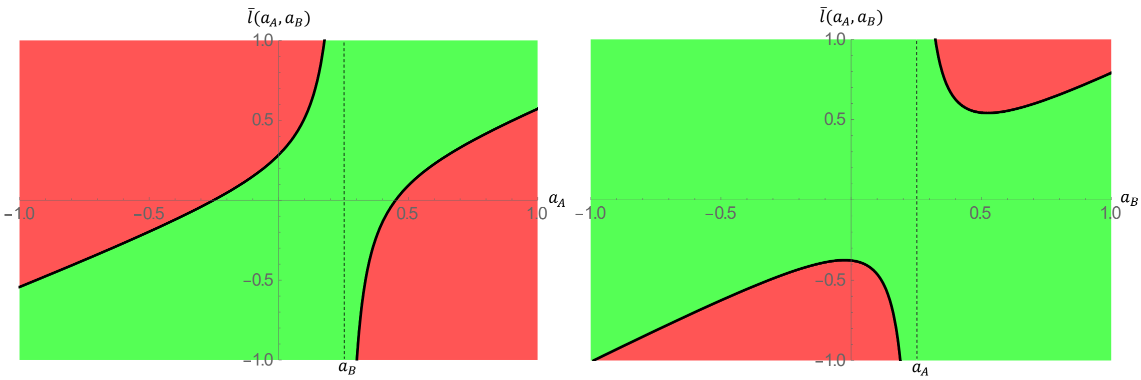


Figure 1: Black curves show the critical $\bar{l}(a_A, a_B)$, for $p_A = 0.8$, $p_B = 0.6$, $s = 0.5$, $\Delta v = 0.1$. The left panel holds $a_B = 0.25$ fixed and varies $a_A$. The right panel holds $a_A = 0.25$ fixed and varies $a_B$. For a given policy pair, $1/2$ times the height of a region equals the winning probability for $A$ (green) and $B$ (red).

The following proposition describes the best response correspondences and the equilibria under different timing assumptions. Figure 2 illustrates the results.

**Proposition 1.** *Assume that $\Delta v > 0$, $\Delta p > 0$, and Inequality* (9) *holds. (i) Candidate $A$'s best response correspondence is a closed interval that includes $a_A = a_B$ (the striped green area in Figure 2). $B$'s best response correspondence is single* *valued except at the discontinuity point* $a_A^* \equiv (1 - p_B)s$*, where there is a downward jump (the red curves in Figure 2). (ii) As follower, A wins with certainty, regardless of B's choice. As leader, A chooses* $a_A = a_A^*$ *and B wins with positive probability, which is less than A's winning probability. (iii) The candidates' best response correspondences are disjoint, so there exists no pure strategy Nash equilibrium under simultaneous moves.*

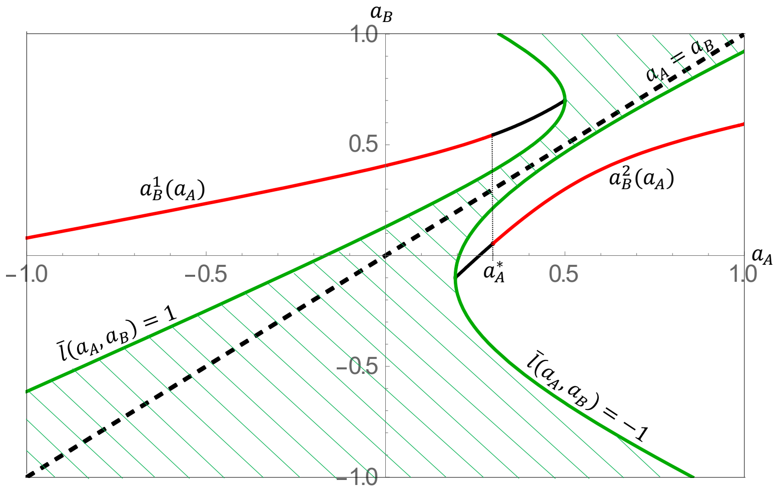


Figure 2: Best response correspondences, for $p_A = 0.8$, $p_B = 0.4$, $s = 0.5$, $\Delta v = 0.1$. Candidate $A$ wins with probability 1 for policies in the green striped area. The red curves show $B$'s best response correspondence, which is discontinuous and multi-valued at $a_A = a_A^*$. On these curves, $B$ has positive probability of winning.

Intuitively, $A$ wins with certainty when following $B$'s announcement, simply by choosing a position close to $B$'s. Inequality (9) ensures that at $A$'s optimal action as leader, $a_A = a_A^*$, $B$ wins with positive probability by following optimally. Because $a_A = a_A^*$ minimizes $B$'s winning probability, $B$ has a still higher probability of winning for any other choice of $a_A$. Therefore, $B$'s best response correspondence lies outside the set where $A$ wins with certainty; that set is $A$'s best response correspondence. This disjointness of best response correspondences implies that a pure strategy Nash equilibrium does not exist.

Part (ii) notes that $A$'s winning probability is greater than $B$'s, even when $A$ leads. Thus, the intrinsic electoral strength associated with $\Delta v > 0$, $\Delta p > 0$ more than offsets a loss in the second-mover advantage.

If the stronger candidate $A$ leads, their optimal action $a_A^* = (1 - p_B)s$ corresponds to the policy platform $x_B = 0$, the policy that maximizes the *other* candidate's appeal to the expected median voter (if $B$ were the sole candidate).[15] Because $A$ is more competent, the policy platform $x_A$ that corresponds to $a_A^*$ is distorted towards the status quo policy $s$, as can easily be verified. As follower, $B$ wants to differentiate themselves from $A$. As leader, $A$ takes advantage of this incentive by choosing the platform corresponding to $x_B = 0$, thereby inducing $B$ to move away maximally from what would otherwise be $B$'s favored platform.

## 2.2 Two limiting cases

Here we consider two limiting cases: (i) a simple case where $\Delta v > 0$ with $\Delta p = 0$ and (ii) the more complex situation where $\Delta v = 0$ with $\Delta p > 0$. These cases highlight the differing roles of competence and valence. The first case is closer to existing literature, but we consider it under different timing regimes.

**Remark 1.** When $\Delta v > 0$ with $\Delta p = 0$, the equilibrium is qualitatively the same as in Proposition 1 (which assumes $\Delta v > 0$ with $\Delta p > 0$). In the special case where $p_A = p_B = 1$, the equilibrium does not depend on the status quo policy; in this case, $A$ chooses the political center $a_A^* = 0$ when it leads.

The anti-coordination structure of the game remains even when $\Delta p = 0$. As noted above, previous models have been criticized because their equilibria do not depend on the status quo, and in that respect are independent of history. The Remark shows that in a more general model the status quo $s$ does affect the equilibrium unless both candidates are perfectly competent ($p_A = p_B = 1$).

The second limiting case eliminates $A$'s valence advantage (setting $\Delta v = 0$) but retains its competence advantage ($\Delta p > 0$). We first describe the outcome in the following Proposition, and then use Figure 3 to provide the intuition for it. Recall that $\alpha$ is the exogenous probability that $A$ wins in the event of a tie.

**Proposition 2.** *(i) In the electoral competition game with $\Delta v = 0$ and $\Delta p > 0$, there exists a pure strategy Nash equilibrium under simultaneous moves if and only if $s = 0$ and $\alpha = 1/2$. (ii) If $A$ leads they choose $a_A = a_A^*$, and $B$ wins with positive probability. (iii) If $B$ leads they choose $a_B = s$, and $B$ wins with positive*

[15] Conditional on $B$ being able to implement its announcement, $x_B = 0$ is the ideal point for the expected median voter. If $B$ is not able to implement its announcement, that announcement does not affect the outcome.

*probability unless $\alpha = 1$. (iv) Each player obtains a higher winning probability as follower than as leader (second-mover advantage).*

**Remark 2.** While all statements in Proposition 2 remain valid, a Nash equilibrium fails to exist (in a strict sense) under sequential moves for some parameter values: (i) $A$ leads, $s = 0$, and $\alpha > 1/2$; (ii) $B$ leads, $s \neq 0$, and $\alpha < \frac{1+|s|}{2}$. In each of these cases, the follower "shades" the leader's action choice, which means that they choose an action arbitrarily close to the leader's action.[16]

The important difference, relative to the case where $\Delta v > 0$, is that $B$ wins with positive probability also as the leader, unless $\alpha = 1$ (Proposition 2.iii). However, $B$ still does better as the follower, and $A$'s behavior as leader is unchanged. Apart from a parameter set of measure zero ($s = 0 \wedge \alpha = 1/2$, Proposition 2.i), there is still no pure strategy Nash equilibrium under simultaneous moves.

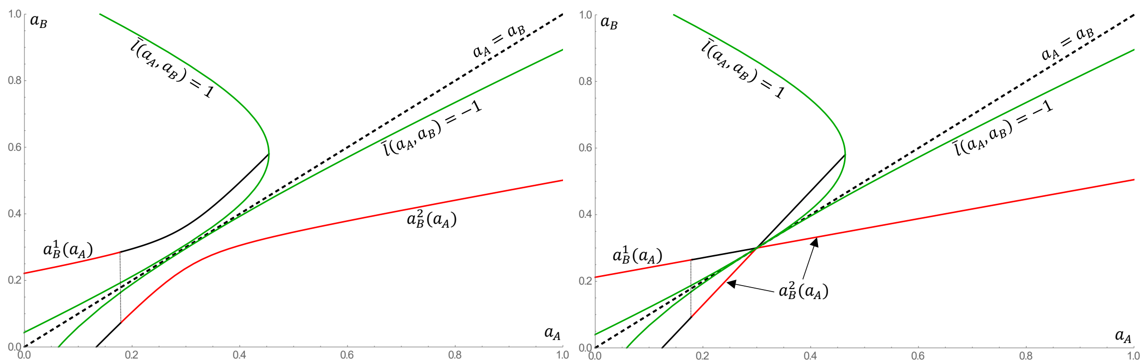


Figure 3: Combinations of $a_A$ and $a_B$ where $\bar{l}(a_A, a_B) = 1$ resp. $\bar{l}(a_A, a_B) = -1$ (green curves), and $B$'s best response correspondence (red curves), for $p_A = 0.8$, $p_B = 0.4$, $s = 0.3$. Left panel: for $\Delta v = 0.01$. Right panel: for $\Delta v = 0$.

Figure 3 illustrates this situation, using $s = 0.3$. The left panel corresponds to $\Delta v = 0.01$ and the right panel corresponds to $\Delta v = 0$. The green curves in both panels show the loci where $\bar{l} = \pm 1$. These are tangent to the 45° line (and thus to each other) at $a_A = a_B = s$ for $\Delta v = 0$. The red curves in both panels show the two arms of $B$'s best response correspondence, with the discontinuity at $a_A^* = (1 - p_B)s$. The left panel, corresponding to the case in Proposition 1, shows these arms bounded away from the 45° line, as is also the case in Figure 2. The

[16]This is well-known from other games, such as Bertrand competition with constant but asymmetric marginal costs. Tirole (1988, p.234) uses a simple characterization of the outcome by allowing the firm with lower marginal cost to "undercut" the competitor's price marginally.

right panel shows the lower arm tangent to the 45° line and to the two $\bar{l}$ loci at $a_A = a_B = s$.[17] This difference causes the qualitative changes described above.

At any $a_A = a_B \neq s$, $A$ wins with certainty. At such a point, the candidates implement the same expected policy, but the variance of the outcome is smaller under the more competent candidate $A$ (see Footnote 11). The median voter then prefers the more competent candidate. This advantage disappears at $a_A = a_B = s$, so here $A$ wins with the exogenous probability $\alpha$.

It appears from the figure that $a_A = a_B = s$ is a pure strategy Nash equilibrium. However, if for example $A$ takes $a_B = s$ as given, $A$ has the option of matching $B$ and winning with probability $\alpha$, or "shading" $B$'s choice by adopting a position close to but not equal to $s$. $B$ has the same opportunity if they take $a_A = s$ as given. The proof of Proposition 2 (see also Remark 2) shows that one of the two agents always has an incentive to defect from $a_A = a_B = s$ unless $s = 0$ and $\alpha = 0.5$, the only parameter combination at which there is a pure strategy Nash equilibrium.

$A$'s behavior as leader (Proposition 2.ii) is the same as when $\Delta v \neq 0$, but $B$'s strategic situation as leader is different (Proposition 2.iii). If $B$ were to adopt any position other than $s$, $A$ wins with certainty by matching that position. However, by choosing $a_B = s$, $B$ knows that $A$ will choose either to match or shade its position. In either case, $B$'s payoff is positive unless $\alpha = 1$. A comparison of $B$'s payoffs as follower or leader shows that $B$ is then always better off as follower.

# 3 Continuous time framework

We introduce a two-player continuous time framework. Later we embed the static electoral competition game from Section 2 into this framework to endogenize the timing of candidates' policy announcements.[18] In the continuous time setting, where there is no "first" subgame after the current time, agents who move at the same time might do so simultaneously or sequentially (Simon and Stinchcombe, 1989). In the former case, neither agent can condition their action on their rival's action. In the latter case, this conditioning is possible.

[17] The tangency occurs on the lower arm because in this figure $s > 0$. For this reason, the figure shows only the positive quadrant.

[18] The framework has many other applications, shown in our companion paper Karp et al. (2024). That paper emphasizes the relation between discrete and continuous time games, and explains the technical assumptions listed below.

In the electoral competition game, like many others, the lack of a pure strategy equilibrium is due to the static assumption that *requires* agents to move at the same time. Sometimes, analytic convenience is the only reason for that requirement. More plausibly, agents can choose when to move, and can do so simultaneously or sequentially. The continuous time setting allows for these possibilities.

We first describe the game and introduce the notation. The next subsection describes a "restricted game" that provides a link between our static and continuous time frameworks. We then present our major result, providing conditions under which there is a unique pure strategy sequential move equilibrium.

## 3.1 Model description and notation

Each of two players, $i \in \{1, 2\}$, can move at most once in the time interval $[0, 1)$. Player $i$ has a default action, $\omega_i$, which we can think of as "waiting". The agent can also select an action $a_i$ from a compact subset of $\mathbb{R}^n$ (e.g., an interval) $A_i^-$. Player $i$'s full action set is thus $A_i = A_i^- \bigcup \omega_i$. If player $i$ moves at $t_i \in [0, 1)$, selecting $a_i$, that action remains fixed after $t_i$. If $i$ never moves, we set $t_i \equiv 1$ and $a_i \equiv \omega_i$. Note, however, that time $t = 1$ is *not* part of the game.

A *decision node* is a time $t \in [0, 1)$, combined with all information about the history, $h$, at $t$. At any node, either one player or neither player has moved; the game is over once both have moved. The history at $t$ is thus either $h = \emptyset$ (nobody has moved), or player $j$ has moved at $t_j \leq t$, implementing action $a_j$. A *strategy*, denoted $f_i(t, h)$, instructs a player who has not previously moved whether to move and what action to take. We assume that players use pure strategies regarding their timing decisions. Section 3.2 explains why they do not mix over actions.

A "discontinuity point" in $f_i$ is a decision node where either the player switches between the default action $\omega_i$ and some $a_i \in A_i^-$, or where $i$ switches discontinuously between two different actions in $A_i^-$. We adopt:

**Assumption 1.** For every history $h$ of the game and $i = 1, 2$, $f_i(t, h)$ contains at most a finite number of discontinuity points and is right-continuous in $t$.

For a subgame beginning at $t$ at which neither player has previously moved, we define the time $t_i^m(t) \in [t, 1]$ where player $i$ plans to move according to his strategy $f(t, \emptyset)$, conditional on the rival not having previously moved, and an associated action $a_i^m(t) \in A_i$ that the player implements at $t_i^m(t)$. If $i$ plans to wait at all

$t \in [0,1)$, provided that the other player also does not move, then $t_i^m(t) = 1$ and $a_i^m(t) = \omega_i$. Players who follow their strategies move simultaneously at $t \in [0,1)$ if and only if $t_i^m(t) = t_j^m(t) = t$.

For a subgame at $t \in [0,1)$ where $j$ has already moved, $i$'s strategy $f_i(t,h)$ consists of functions $t_i^s(t|t_j,a_j) \in [t,1]$ and $a_i^s(t|t_j,a_j) \in A_i$, determining the follower's response and time of response. The superscript $s$ indicates that $i$ is the second-mover. If $i$ follows $j$'s move immediately, then $t_i^s(t_j|t_j,a_j) = t_j$. If $i$ plans not to make any move after observing $j$'s move, then $t_i^s(t|t_j,a_j) = 1$ and $a_i^s(t|t_j,a_j) = \omega_i$.

This description of strategies emphasizes that agents might move at the same point in time either simultaneously or sequentially. In the first case, neither agent can condition their action on the other agent's action. In the second case, the follower can condition their action on the leader's action and on its timing. The equilibrium actions may differ in these two cases. Assumption 1 implies that $t_i^m(t)$, $a_i^m(t)$, $t_i^s(t|t_j,a_j)$, and $a_i^s(t|t_j,a_j)$, are right-continuous and have a finite number of discontinuities in $t$. Therefore, there can be no sequence of subgames over which the leader-follower roles switch infinitely many times. This rules out a particular type of preemption, and is used in the proof of Theorem 1.

Assumption 1 does not specify how player $i$'s strategy as a follower varies with $t_j$ and $a_j$, the timing and action choice of the leader. We thus adopt:

**Assumption 2.** The functions $t_i^s(t|t,a_j^m(t))$ and $a_i^s(t|t,a_j^m(t))$ are right-continuous in $t$ over any interval where player $j$ plans to lead (i.e., where $t_j^m(t) = t$ holds).

Let player $i$'s *payoff* be $\Pi_i(t_i,a_i,t_j,a_j)$. If neither player moves in $[0,1)$, then player $i$ obtains a fixed endgame payoff $E_i \equiv \Pi_i(1,\omega_i,1,\omega_j)$.

**Assumption 3.** $\Pi_i(t_i,a_i,t_j,a_j)$ is *bounded* and *continuous.*

Continuity of players' payoffs in their timing and action choices seems natural in many applications where players' action sets are infinite.

If player $i$ reaches the decision node at $t \in [t_j,1)$, after an observed prior move by player $j$, $(t_j,a_j)$, and $i$ behaves optimally at $t$, $i$ obtains the payoff

$$\Pi_i^s(t|t_j,a_j) \equiv \max_{t_i^s \in [t,1],\, a_i^s \in A_i} \Pi_i(t_i^s,a_i^s,t_j,a_j). \tag{10}$$

The solution to this optimization problem consists of the functions $t_i^{s*}(t|t_j,a_j)$ (the optimal time to move) and $a_i^{s*}(t|t_j,a_j)$ (the optimal action to take). If the solution is not unique, we assume that both players know which pair of maximizers (compatible with Assumption 2) the follower chooses.

Conditional on leading at time $t \in [0,1)$, whether or not it is optimal to do so, we define the leader's (player $i$'s) maximum payoff as

$$L_i(t) \equiv \max_{a_i \in A_i^-} \Pi_i(t, a_i, t_j^{s*}(t|t, a_i), a_j^{s*}(t|t, a_i)). \tag{11}$$

We denote player $i$'s optimal action, conditional on leading at time $t$, as $a_i^L(t)$.[19] We again assume that the maximizer $a_i$ in Equation (11) is unique, or if not, that both players anticipate which maximizer $i$ will choose if this player leads at time $t$.

If player $j$ leads at time $t \in [0,1)$, implementing their optimal action $a_j^L(t)$, then the follower, player $i$, obtains the payoff

$$F_i(t) \equiv \Pi_i^s(t|t, a_j^L(t)). \tag{12}$$

This definition embeds player $i$'s optimal behavior as the follower, via Equation (10). Note that $F_i(t)$ is not the payoff $i$ gets when moving *at* $t$, but rather the payoff from following a move by $j$, taking place at time $t$, optimally (irrespective of whether this has $i$ move immediately after $j$, at a later point in time, or not at all).

## 3.2 A bridge between the static and dynamic games

There are many plausible ways to generalize a static game to make it dynamic, but there is an obvious way to specialize a dynamic game to make it static: For all $t$, give players the same payoff and action sets as in the dynamic game but require that they move simultaneously at $t$. We refer to this as the "restricted game" at $t$.[20] Our assumptions above ensure that there exists a Nash equilibrium to this game (Zhou et al., 2011). If there exist both pure and mixed strategy equilibria, we assume that players coordinate on a (particular) pure strategy Nash equilibrium, resulting in player $i$'s (expected) equilibrium payoff $N_i(t)$.

The dynamic game allows players to wait at $t$, whereas the restricted game does

[19] Note the difference between the functions $a_i^m(t)$ and $a_i^L(t)$. The former describes an arbitrary strategy of player $i$, stating the action that the player implements at the *next* point in time when this player plans to move, $t_i^m(t)$. The latter function describes the optimal action for player $i$ to implement *at* $t$, conditional on leading at this time.

[20] This restricted game is of interest only if neither player has previously moved.

not. The restricted game thus provides a simple means of obtaining sufficient conditions to rule out the existence of a simultaneous move equilibrium in the dynamic game. We merely need to determine whether the restricted-game Nash equilibrium constitutes an equilibrium in the dynamic game. That is, we ask whether the restriction to simultaneous moves would bind in the dynamic game. If the restriction does bind, then there are no simultaneous move equilibria in the dynamic game.[21] The following Assumption states that the restriction binds:

**Assumption 4.** For every $t \in [0, 1)$ in the dynamic game, at subgames where neither player has previously moved, at least one player strictly prefers to wait at $t$ rather than to use their restricted-game Nash equilibrium strategy, given that the other player adopts their restricted-game Nash equilibrium strategy at $t$.

This assumption might be satisfied in at least two ways. First, waiting might increase a player's payoff independently of the actions taken, possibly due to exogenous changes. Second, there may exist no pure strategy simultaneous move equilibrium in the restricted game, leaving only a mixed strategy equilibrium. If player $i$ were to use its restricted-game mixed strategy in the dynamic game at $t$, then $j$ strictly prefers to deviate from its own restricted-game mixed strategy. Player $j$ does better by waiting to observe the outcome of $i$'s strategy and then responding optimally. In the continuous time setting, $j$ incurs no cost from this "delay". Thus, given that agents do not randomize over their timing decisions, they do not mix over actions.[22]

We are interested in the second reason. For the static electoral competition game that motivates our use of the continuous time setting, the only simultaneous move equilibrium (in most cases) is in mixed strategies. We noted above that there are many ways to generalize a static game by embedding it into a dynamic setting, e.g., different types of discounting or exogenous changes. However, a minimal requirement for this generalization is that for all $t \in [0, 1)$ the restricted

[21] If there was such an equilibrium in the dynamic game, then it would also be a Nash equilibrium in the restricted game.

[22] Equilibria in which players randomize over the timing of their moves by leading with some hazard rate are easy to rule out, as shown by Hendricks et al. (1988). This merely requires an asymmetry between players. Below, we introduce a time $\tau_i$ for player $i$, which is the latest time at which this player is willing to lead, conditional on the other player not leading after this time. Provided that $\tau_1 \neq \tau_2$, no mixed strategy equilibria of the "hazard rate type" can exist in our setting. This is implied by our arguments in the proof of Theorem 1 (below), and conforms with results of Hendricks et al. (1988) for a simpler setting where players can only choose the timing of their moves (and not their actions).

game payoff, $\Pi_i(t, a_i, t, a_j)$, is isomorphic[23] to the payoff in the original static game, $\pi_i(a_i, a_j)$; and the action sets $A_i^-$ are the same in the two settings.

**Proposition 3.** *Suppose that for all* $t \in [0, 1)$ *the payoffs* $\Pi_i(t, a_i, t, a_j)$ *and* $\pi_i(a_i, a_j)$ *are isomorphic, the action sets* $A_i^-$ *in the two games are the same, and there exist no pure strategy Nash equilibria in the static game. Then there exist no SPNE in the dynamic game where players move simultaneously.*

Proposition 3 implies that when the static game has only mixed strategy equilibria, then Assumption 4 is satisfied. With that and the previous technical assumptions, we can sometimes identify a unique SPNE with sequential moves in the dynamic game. The profession is so habituated to mixed strategy equilibria that we may forget that these sometimes arise from the assumption that agents must move simultaneously. That assumption might be adopted for tractability, without a compelling economic rationale. However, using a continuous time formulation of the game, we discover that in some cases the dynamic game is both more tractable, and its predictions more plausible than its static counterpart. Section 4 illustrates this claim using the electoral competition model from Section 2.

## 3.3 Main result

We hereafter consider games with a "second-mover advantage", as in the electoral competition game. Player $i$ has a strict second-mover advantage at $t$ if, assuming that the other player moves at $t$, $i$ strictly prefers to wait until $t$, rather than lead shortly before $t$. More formally:

**Definition 1.** *Player* $i$ *has a* strict second-mover advantage *at* $t \in (0, 1)$ *if there exists some* $\delta > 0$ *such that* $F_i(t) > \sup_{t' \in [t-\delta, t]} L_i(t')$*, and at* $t = 0$ *if* $F_i(0) > L_i(0)$.

Figure 4 illustrates two cases that make it easy to understand both our notation and the role of "patience" in determining which player leads in equilibrium. In both panels, player 1 (but perhaps not player 2) has a strict second-mover advantage at all $t \in [0, 1)$. For both players, the payoff from leading at $t$, $L_i(t)$, is single-peaked with $i$'s ideal time located at $\bar{t}_i \geq 0$. The possible non-monotonicity of $L_i(t)$ accommodates non-strategic reasons for a player to delay leading. For example, the timing of an advertisement or electoral campaign might be important.

[23]Isomorphic means that the location of minima, maxima, signs of slopes, curvatures etc. are the same in $\pi_i$ and $\Pi_i$ for identical values of $a_i$ and $a_j$.

In addition, for each player $i$ there is a first decision node, a critical value denoted $\tau_i$, beyond which $i$ does not want to lead provided that the other player also does not lead for the rest of the game. In both panels, $\tau_2 > \tau_1$. The main difference between the panels is that $\bar{t}_1 > \bar{t}_2$ in panel A, but $\bar{t}_2 > \bar{t}_1$ in panel B.

Here, patience has three dimensions. (i) A second-mover advantage gives an agent a reason to delay moving. (ii) An earlier critical time, $\tau_i$ means that from an earlier time player $i$ prefers to wait (rather than lead) until the end of the game. (iii) A later ideal time to lead, $\bar{t}_i$, creates a third reason for patience. In Figure 4A, player 1 is more patient in both the second and third dimensions (and possibly also in the first), so it is not surprising that for this configuration player 2 leads in the unique SPNE. In contrast, in Figure 4B player 1 is more patient in the second dimension but player 2 is more patient in the third one.[24] Here, the identity of the leader depends on the relative strength of the two types of patience.

We now define three pieces of notation. Let

$$\tau_i \equiv \inf\{t \in [0,1) : L_i(t') < E_i \text{ for all } t' \geq t\} \text{ ("latest time for } i \text{ to lead")}, \tag{13}$$

$$\bar{t}_i \equiv \underset{t\in[0,1)}{\operatorname{argmax}}\, L_i(t), \text{ ("best time for } i \text{ to lead")}, \tag{14}$$

$$\hat{t}_i \equiv \sup\{t \in [0, \bar{t}_j) : L_i(t) \geq F_i(\bar{t}_j)\} \text{ ("latest time for } i \text{ to preempt } j\text{")}. \tag{15}$$

If there is no $t \in [0,1)$ such that $L_i(t') < E_i$ for all $t' \geq t$, we set $\tau_i = 1$.[25] We further assume that the maximum in Definition (14) exists and is unique.[26] The value $\hat{t}_i$ is the upper boundary of an interval where $i$ prefers to preempt $j$ by leading prior to $j$'s optimal time to lead, $\bar{t}_j$, rather than wait for $j$ to lead (see Figure 4B for illustration). If no such time exists, we set $\hat{t}_i = 0$.

We now collect conditions used in Theorem 1:

**Condition 1.** (Conditions relevant for Theorem 1)

(i) At every $t \in [\bar{t}_2, \tau_2)$, player 1 has a strict second-mover advantage and $L_2(t)$ is strictly decreasing over this interval; in addition, $\tau_2 > \tau_1$.

(iia) $\sup_{t\in[0,\bar{t}_2)} L_1(t) < F_1(\bar{t}_2)$.

[24] Theorem 1 does not require all of the features (e.g., continuity everywhere, or that $F_1(t) > L_1(t)$ holds for *all* $t \in [0,1)$) shown in Figure 4.

[25] This is for instance the case if $L_i(t') \geq E_i$ for all $t \in [0,1)$.

[26] We thus rule out the (knife-edge) case where $L_i(t)$ has a downwards discontinuity exactly at the point where it attains its highest value.

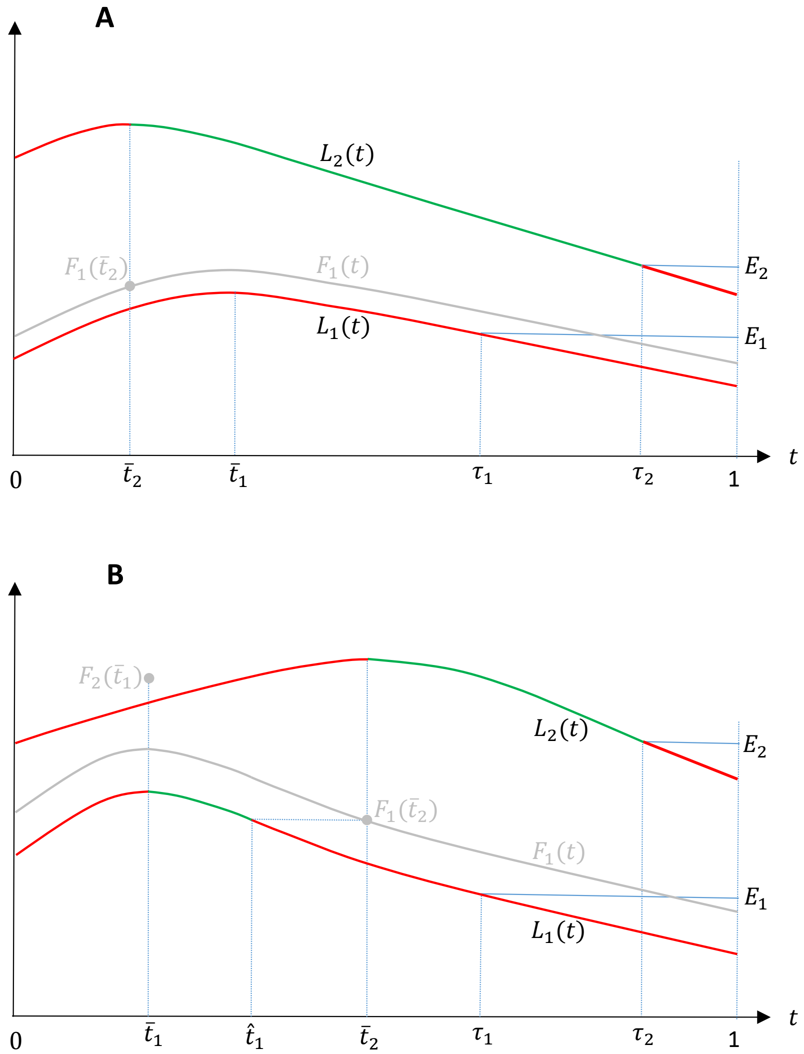


Figure 4: Payoff functions $L_i(t)$ and $F_i(t)$ in stylized examples with a second-mover advantage for player 1, and $L_2(t)$ monotonically decreasing for $t > \bar{t}_2$ (player 2's ideal point for leading); **panel A**: example where $\bar{t}_1 > \bar{t}_2$; **panel B**: example where $\bar{t}_2 > \bar{t}_1$; Green color: player $i$ leads at $t$ in unique SPNE, Red: player $i$ waits (see Theorem 1).

(iib) $\sup_{t\in[0,\bar{t}_2)} L_1(t) > F_1(\bar{t}_2)$, and $\sup_{t\in[0,\bar{t}_1)} L_2(t) < F_2(\bar{t}_1)$.

(iii) At every $t \in [\bar{t}_1, \hat{t}_1)$, player 2 has a strict second-mover advantage, and $L_1(t)$ is strictly decreasing over this interval.

Condition 1.iia implies that player 1 does not want to preempt player 2 by leading prior to $\bar{t}_2$, 2's optimal time to lead. The first part of Condition 1.iib states that player 1 would like to preempt player 2 by leading before $\bar{t}_2$. The second part of this condition states that player 2 is willing to accommodate this preemption. Condition 1.iii mirrors Condition 1.i, with the roles of the two players reversed. The proof of Theorem 1 uses the observation that under Conditions 1.iib and 1.iii, the subgames in the interval $t \in [\bar{t}_1, \bar{t}_2)$ are isomorphic to those in the interval $t \in [\bar{t}_2, 1)$, with the roles of the players reversed.

The following condition assures *uniqueness* of the SPNE:[27]

**Condition 2.** (Conditions for uniqueness in Theorem 1)

(i) For any $t \in [0, 1)$, $t_j \leq t$, and $a_j \in A_j^-$, the maximizers of Equation (10) are unique.

(ii) For any $t \in [0, 1)$, the maximizer of Equation (11) is unique.

If one player has previously moved, the other agent faces a standard decision problem. In Theorem 1, it is understood that a follower responds optimally to any prior move of the other player; and a player who leads at time $t$ uses their best leader's action, $a_i^L(t)$ (Section 3.1). In Figure 4, the curve $L_i(t)$ is green if (according to Theorem 1) $i$ leads at time $t$, and red if $i$ waits.

**Theorem 1.** *Suppose that Assumptions 1–4, as well as Condition 1.i hold. Consider subgames starting at time $t \in [0, 1)$, such that nobody has previously moved:*

*(i) (See Figure 4.) In subgames starting at any $t \in [\bar{t}_2, 1)$, there is a SPNE: player 1 waits at decision nodes $t \in [\bar{t}_2, 1)$; player 2 leads at nodes $t \in [\bar{t}_2, \tau_2)$, and waits at $t \in [\tau_2, 1)$.*

*(ii) (See panel A in Figure 4.) If Condition 1.iia holds, there is also a SPNE for subgames starting at $[0, \bar{t}_2)$: both players wait at nodes $t \in [0, \bar{t}_2)$. (Players' behavior at later decision nodes is as under (i).)*

*(iii) (See panel B in Figure 4.) If Conditions 1.iib and 1.iii hold, there is a SPNE for subgames starting at $[0, \bar{t}_2)$, with the strategies: player 1 leads at decision nodes $t \in [\bar{t}_1, \hat{t}_1)$ and waits at $t \in [0, \bar{t}_1) \bigcup [\hat{t}_1, \bar{t}_2)$; player 2 waits at decision nodes $t \in [0, \bar{t}_2)$. (Players' behavior at later decision nodes is as under (i).)*

*If, additionally, Condition 2 holds, then the respective SPNE is **unique**.*

Online Appendix B.3 provides an intuition for Theorem 1.

[27] If this condition does not hold, then (as in Section 3.1) we assume that both players anticipate which maximizer is selected. Then the SPNE is unique conditional on that choice.

### 3.4 Extension: Non-monotonicity of $L_i(t)$

Condition 1(i), and thus Theorem 1, requires that over particular intervals, $L_i(t)$ is strictly decreasing. Here we show how to restore uniqueness in one case where this monotonicity fails. The top graphs in Figure 5 show two different types of non-monotonic functions $L_i(t)$. In panel A, $L_i(t)$ has an upwards jump, while in panel B it has an increasing segment. For both panels, between $t'$ and $t''$ this player prefers to wait rather than lead, conditional on the other player also waiting. The functions $\bar{L}_i(t)$, shown at the bottom, are obtained by "ironing" (or "flattening") $L_i(t)$: between $t'$ and $t''$, $\bar{L}_i(t)$ is constant, equal to the higher value at the discontinuity point (panel A), or at the local maximum (panel B).

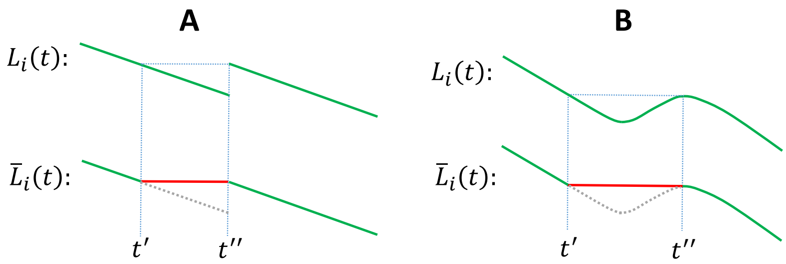


Figure 5: $L_i(t)$ non-monotonic; panel A (top part): $L_i(t)$ has an upwards discontinuity at $t''$; panel B (top): $L_i(t)$ has an increasing segment; bottom part of each panel: "modified leader's payoff function" $\bar{L}_i(t)$ (player $i$ waits between $t'$ and $t''$)

The functions $L_i(t)$ and $\bar{L}_i(t)$ have different interpretations. By definition, $L_i(t)$ is the maximal payoff of player $i$, conditional on leading at time $t$, regardless of whether this is optimal for player $i$ at $t$. The function $\bar{L}_i(t)$, on the other hand, incorporates the fact that between $t'$ and $t''$, $i$ prefers to wait, conditional on $j$ also waiting. The red segment in each panel indicates this interval. Outside of the interval $(t', t'')$, $\bar{L}_i(t)$ has the same interpretation as $L_i(t)$.

When the monotonicity of $L_i(t)$ stated in Condition 1(i) fails over $(t', t'')$, a sufficient alternative condition for uniqueness is that $j$ does not want to lead over this interval given that the alternative is to wait until $t''$ and enjoy the second-mover advantage: $j$ does not want to preempt $i$. This sufficient condition is

**Condition 3.** (Condition relevant for Corollary 1)

$\sup_{t\in[t',t'']} L_j(t) < F_j(t'')$.

This alternative to Condition 1.i is useful in the next section, so we state:[28]

**Corollary 1.** *Suppose that Condition 1.i fails, so that a non-monotonicity in $L_i(t)$ gives rise to the interval $[t', t'']$ over which the "ironed" function $\bar{L}_i(t)$ is constant. Moreover, Condition 3 is satisfied. The unique SPNE is as described in Theorem 1, except that both agents wait over the interval $[t', t'']$.*

Remark 3 in Section 4 considers a case where both, Conditions 1.i and 3, fail.

# 4 Electoral competition in continuous time

Here we embed the electoral competition model from Section 2 into a continuous time setting, thus allowing candidates to decide when to announce their platform. Section 3.2 notes that a minimal requirement in using a static game to construct a continuous time game is that for all $t \in [0, 1)$ the restricted game payoff, $\Pi_i(t, a_i, t, a_j)$, is isomorphic to the payoff in the original static game, $\pi_i(a_i, a_j)$; and the action sets $A_i^-$ are the same in the two settings. We assume that this condition is met. However, to complete the description of the dynamic game we need to introduce time dependence of payoffs and also extend the domain of the static game payoff functions to include $\omega_i$ ("waiting").

We adopt the following formulation: For $a_i \in A_i^-$, $t_i < 1$, and $i \in \{A, B\}$, (i.e., when both candidates announce a policy platform)[29]

$$\Pi_i(t_i, a_i, t_j, a_j) = (1 - t_i)\pi_i(a_i, a_j) - c_i. \tag{16}$$

In the static game $\pi_i(a_i, a_j)$ is $i$'s probability of winning when the candidates announce $(a_i, a_j)$. In the dynamic game, candidate $i$ incurs a fixed cost $c_i$ from entering. Note that delaying entry harms the candidate, without benefiting the rival. For example, it may raise the cost of running a campaign, reduce opportunities for fund-raising, or expose the candidate to the risk of being replaced.

[28]The proof of this corollary closely follows the proof of Theorem 1, except that over $[t', t'']$ player $i$ prefers to wait, knowing that $j$ will not preempt it. If the function $L_i(t)$ (rather than the ironed function $\bar{L}_i(t)$) were constant over $[t', t'']$, then there is a (trivial) multiplicity of equilibria: over some interval $i$ is indifferent between leading and waiting.

[29]Note, that the right-hand side depends only on $t_i$, the timing of player $i$'s own move. This formulation provides perhaps the most parsimonious extension to a dynamic setting of our static model. Clearly, more general formulations are available, but our main interest is endogenizing the timing of moves in a simple setting that captures the main features of the underlying electoral competition model.

Let $a_i^L$ be the leader's action choice in the static electoral competition model if $i$ leads. In the dynamic game, if $A$ announces $a_i^L$ at a time $t_i < 1$ at which $j$'s optimal response is to never respond (so that $t_j^{s*}(t_i|t_i, a_i) = 1$), $i$'s payoff is

$$\Pi_i(t_i, a_i^L, 1, \omega_j) = (1 - t_i)\bar{\pi}_i - c_i, \tag{17}$$

where $\bar{\pi}_i \leq 1$ is a free parameter. If $\bar{\pi}_i = 1$, then $i$ wins by default. If $\bar{\pi}_i < 1$, then $i$ faces the risk of a late entrant when $j$ chooses not to announce a policy.[30] We assume that leading at $t_i$ with an action different from $a_i^L$ leads to a strictly lower payoff to candidate $i$. Hence, also in the dynamic game, conditional on leading, candidate $i$ always chooses the same action $a_i^L$.[31]

Finally, if neither player moves in $[0, 1)$, then both candidates obtain a payoff of zero, i.e., $E_i = \Pi_i(1, \omega_i, 1, \omega_j) = 0$. And if only player $j$ moves in $[0, 1)$, but player $i$ never moves, again, player $i$ obtains a payoff of zero: $\Pi_i(1, \omega_i, t_j, a_j) = 0$ for all $a_j \in A_j^-$ and $t_j < 1$.

We assume that either (i) $\Delta v > 0$ and $\Delta p \geq 0$, or (ii) $\Delta v = 0$ and $\Delta p > 0$, so candidate $A$ is the stronger candidate in the static game. For the second case, we exclude $s = 0$ (see Remark 2 in Section 2.2) to ensure that there does not exist a pure strategy Nash equilibrium in the static game under simultaneous moves in either case, and that $A$ always chooses $a_A^*$ as leader. We also assume that Condition (9) holds. If $-1 \leq \bar{l}(a_A, a_B) \leq 1$, where $\bar{l}(a_A, a_B)$ is given by Equation (7), $A$'s winning probability is (see Section 2)

$$\pi_A(a_A, a_B) = \frac{1}{2} \cdot \begin{cases} 1 - \bar{l}(a_A, a_B), & \text{if } a_A > a_B \\ 1 + \bar{l}(a_A, a_B), & \text{if } a_A < a_B. \end{cases}$$

For $\Delta a$ sufficiently close to zero and $\Delta v > 0$, the condition $-1 \leq \bar{l}(a_A, a_B) \leq 1$ is not satisfied, and $A$ wins with probability one (see Figure 2 in Section 2). Under

[30] A candidate's decision not to adopt a policy before $t = 1$ constitutes an implicit withdrawal, which does not necessarily benefit the rival. For example, Kamala Harris was nominated by the Democrats following Joe Biden's withdrawal from the 2024 US presidential election.

[31] We thus assume that $\pi_i(a_i, \omega_j)$ has the same maximizer as $\pi_i(a_i, a_j^s(a_i))$. Note, that "entry deterrence" is not a viable strategy for the leader, because the leader's action already minimizes the follower's election probability. Karp et al. (2024) analyze a price competition game where entry deterrence plays a role (see also Online Appendix B.4 for a related discussion). Caruana and Einav (2008) analyze a dynamic game in which commitment arises endogenously, although players can change their actions over time; they use this framework to study entry deterrence.

a tie, which occurs only if $\Delta v = 0$, $\Delta p > 0$ and $a_A = a_B = s$,[32] candidate $A$ wins with probability $\alpha$. $B$'s winning probability is $\pi_B(a_A, a_B) = 1 - \pi_A(a_A, a_B)$.

We use $\pi_i^F$ and $\pi_i^L$ to denote $i$'s equilibrium winning probability as follower or leader in the static game[33]. We have

$$\text{(i) } \pi_j^F = 1 - \pi_i^L \text{ , (ii) } \pi_i^F > \pi_i^L, \text{ (iii) } \pi_A^L > \pi_B^L \text{ and (iv) } \overline{\pi}_A \geq \overline{\pi}_B. \tag{18}$$

Condition (18).i is a property of probabilities. Condition (18).ii states that both candidates have a second-mover advantage. Condition (18).iii follows from Propositions 1 and 2 and our assumption that $A$ is the stronger candidate. The new condition (18).iv is consistent with the assumption that $A$ is the stronger candidate: $A$ does better than $B$ when the rival never announces a policy, and thus (essentially) drops out of the race. We treat $\pi_i^L, \pi_i^F$, and $\bar{\pi}_i$ as parameters, and use them to define the equilibrium payoffs, conditional on the order of moves, in the dynamic game. We can relate $\pi_i^L$ and $\pi_i^F$ to the parameters of the static game; Online Appendix B.4 contains details.

Using Equations (12) and (17), $i$'s payoff as follower in the dynamic game is

$$F_i(t) = \max\{(1-t)\pi_i^F - c_i, 0\}. \tag{19}$$

Candidate $i$ follows the leader immediately for

$$t < t_i^{crit} \equiv \max\{1 - \frac{c_i}{\pi_i^F}, 0\} \tag{20}$$

and never responds for $t \geq t_i^{crit}$.[34] Thus, $i$'s payoff, conditional on leading at $t$, is

$$L_i(t) = \begin{cases} (1-t)\pi_i^L - c_i & \text{if } t < t_j^{crit} \\ (1-t)\bar{\pi}_i - c_i, & \text{if } t \geq t_j^{crit}. \end{cases} \tag{21}$$

To rule out trivial cases, we assume that the entry costs are small enough that early in the game, $A$ is willing to lead and $B$ is willing to announce a policy as

[32] This is the only parameter constellation that produces a strictly positive probability of a tie.

[33] For $\Delta v = 0$, when the weaker candidate leads, the stronger candidate may not have a best response as this candidate may prefer to shade the weaker candidate's action choice (see the proof of Proposition 2.iii, and Remark 2 in Section 2.2). However, the limit value for the winning probability of each candidate is still well-defined. Moreover, in Proposition 4 we will show that in the dynamic game it is the stronger candidate who leads in equilibrium.

[34] Assumption 2 (right continuity) breaks the tie at $t = t_i^{crit}$.

the follower: $c_A < \pi_A^L$ and $c_B < \pi_B^F$, so $L_A(0) > 0$ and $F_B(0) > 0$. Additionally, we assume $c_A \leq c_B$.

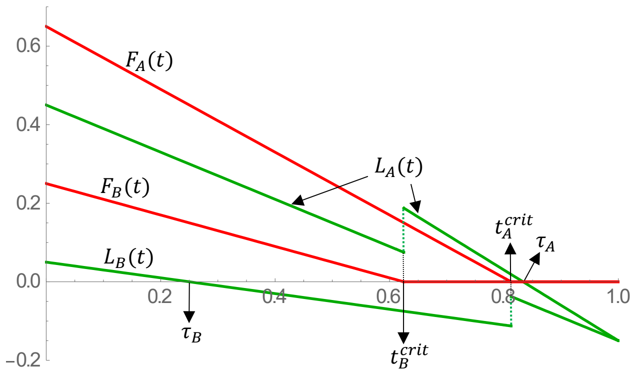


Figure 6: $L_i(t)$ (green) and $F_i(t)$ (red) in the dynamic electoral competition game, for $\pi_A^L = 0.6$, $\pi_B^L = 0.2$, $\overline{\pi}_A = 0.9$, $\overline{\pi}_B = 0.6$, $c_1 = c_2 = 0.15$

Figure 6 illustrates $L_i(t)$ and $F_i(t)$ that satisfy these assumptions. For both candidates, $F_i(t)$ consists of a linearly decreasing segment with $F_i(0) > 0$, followed by a constant segment with the value zero, beginning at $t_i^{crit} < 1$. In addition, $t_A^{crit} > t_B^{crit}$. $L_i(t)$ is piecewise linearly decreasing, with a potential discontinuity at $t = t_j^{crit}$; if $\overline{\pi}_i = \pi_i^L$, the function is continuous.

We need two additional restrictions to apply Theorem 1. The first restriction implies that the weaker candidate $B$ does not lead beyond $t \geq t_A^{crit}$, where the strong candidate $A$ is unwilling to follow. If $B$ were to lead at $t_A^{crit}$ it receives the payoff $(1 - t_A^{crit})\overline{\pi}_B - c_B$. We require that this payoff is negative, i.e., using Definition (20), that

$$c_A\overline{\pi}_B - c_B\pi_A^F < 0, \quad (\text{“}B \text{ does not lead late in the game”}). \tag{22}$$

With this restriction, $B$ never leads at any $t \geq 1 - c_B/\pi_B^L \equiv \tau_B$.[35]

Our second restriction arises if $\overline{\pi}_A > \pi_A^L$; here there is an upward jump in $L_A(t)$ at $t = t_B^{crit}$, as Figure 6 illustrates. In this case, we apply the “ironing” procedure described in Section 3.4, producing the flat interval over $[t_A^\sharp, t_B^{crit}]$ in $A$'s payoff, shown in the graph of $\overline{L}_A(t)$ in Figure 7.[36] As Section 3.4 explains,

[35] There are three possible cases for $\tau_A$. If both segments of $L_A(t)$ intersect the horizontal axis, then $\tau_A$ is the larger of the two intersection points. If there is only one intersection, this point is $\tau_A$. If neither segment intersects the horizontal axis, then $\overline{\pi}_A < \pi_A^L$ (so $L_A(t)$ has a downwards jump at $t_B^{crit}$); here, $\tau_A$ equals the point of discontinuity, i.e., $\tau_A = t_B^{crit}$.

[36] Setting $L_A(t_A^\sharp) = L_A(t_B^{crit})$ yields $t_A^\sharp = 1 - (1 - t_B^{crit})\overline{\pi}_A/\pi_A^L = 1 - c_B\overline{\pi}_A/(\pi_B^F\pi_A^L)$.

here the application of Theorem 1 requires that $B$ does not want to preempt $A$ by leading in this interval. Because $L_B(t)$ is decreasing over this interval, the "no preemption" Condition (3), is $L_B(t_A^\sharp) < 0$, which is equivalent to[37]

$$\pi_A^L(1 - \pi_A^L) > \overline{\pi}_A \pi_B^L \text{ (“}B\text{ does not preempt while }A\text{ waits”).} \tag{23}$$

We summarize these two restrictions in

**Condition 4.** (i) $B$ does not lead when $A$ would not follow: Inequality (22) is satisfied. (ii) $B$ does not preempt while $A$ waits: Inequality (23) is satisfied.

We now apply Theorem 1 to solve the dynamic electoral competition game:[38]

**Proposition 4.** *In the dynamic version of the electoral competition game, under the assumptions and conditions of Theorem 1, together with Condition 4, there is a unique SPNE in which candidate A leads at $t = 0$ and B follows immediately.*

*In subgames where neither player has moved, the strategies are:*

- *(i) if $\overline{\pi}_A \leq \pi_A^L$ or $L_A(t_B^{crit}) \leq 0$ (or both), A leads at $t < \tau_A$, and waits thereafter, while B always waits;*
- *(ii) if $\overline{\pi}_A > \pi_A^L$ and $L_A(t_B^{crit}) > 0$, strategies are the same except that A waits for $t \in [t_A^\sharp, t_B^{crit})$.*

*In subgames where one candidate has moved, the other candidate follows optimally.*

Condition 4.i simplifies the statement and proof of Proposition 4, but the main conclusions remain even without that condition:

**Remark 3.** If Condition 4.i, fails, both players have a first mover advantage in off-path subgames at $t \geq t_A^{crit}$. Therefore, the equilibrium strategies in these subgames can differ from those described in Proposition 4. However, the equilibrium strategies at earlier subgames are unchanged. Therefore, the outcome of the game at $t = 0$ does not change: $A$ leads and $B$ follows immediately.

[37] To establish this equivalence, use $L_B(t_A^\sharp) < 0 \iff c_B > (1 - t_A^\sharp)\pi_B^L$. Using the formula for $t_A^\sharp$ in Footnote 36 we obtain $(1 - t_A^\sharp)\pi_B^L = c_B \overline{\pi}_A \pi_B^L/(\pi_B^F \pi_A^L) < c_B \iff \overline{\pi}_A \pi_B^L/(\pi_B^F \pi_A^L) < 1$. Using $\pi_B^F = 1 - \pi_A^L$, this yields Inequality (23).

[38] Online Appendix B.5 provides a preliminary welfare analysis comparing the efficiency of the sequential equilibrium to a (hypothetical) setting where a planner chooses candidates' policy platforms to maximize the expected welfare of the median voter. Numerical results suggest that the planner chooses policies that are farther apart than the equilibrium policies, thereby catering to more extreme realizations of the median voter's bliss point.

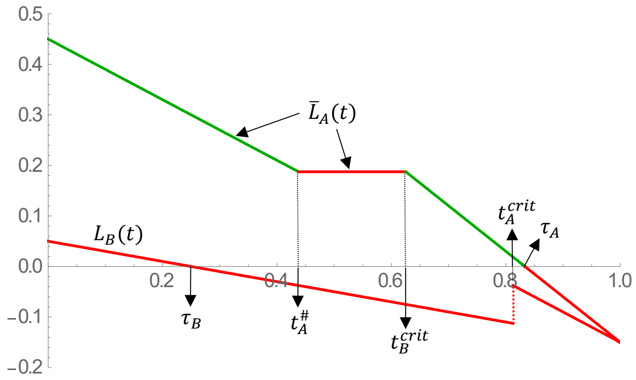


Figure 7: SPNE strategies in the dynamic electoral competition game: $\bar{L}_A(t)$ and $L_B(t)$, green: player leads in subgame beginning at $t$, red: player waits; for $\pi_A^L = 0.6$, $\pi_B^L = 0.2$, $\overline{\pi}_A = 0.9$, $\overline{\pi}_B = 0.6$, $c_1 = c_2 = 0.15$

Figure 7 graphs the candidates' payoffs if they lead, $\bar{L}_A(t)$ and $L_B(t)$, for the case $\overline{\pi}_A > \pi_A^L$. The color coding identifies the candidates' strategies, red indicating "wait" and green indicating "lead". Candidate $B$'s equilibrium strategy is to wait at every subgame where neither player has moved. Candidate $A$ leads at every such subgame before $\tau_A$, except over the interval $[t_A^\sharp, t_B^{crit})$, where $A$ waits. Our key finding is that the stronger candidate $A$ leads, i.e., announces their policy platform before the other candidate. Most importantly, candidates do not announce their platforms simultaneously if they are free to decide when to move.

Figure 7 illustrates a case where both candidates obtain a positive payoff from leading near the beginning of the game. Nevertheless, the stronger candidate leads in the unique equilibrium. The explanation is that the candidates recognize that there are later (off-path) subgames at which the weaker candidate would not lead (because doing so returns a negative payoff), but the stronger candidate is still willing to lead. Reasoning backwards in time, the stronger candidate thus starts leading from an earlier time onwards to avoid inefficient delay, and ends up leading immediately at $t = 0$. Both candidates would prefer to follow rather than lead, but only the weaker candidate enjoys their second-mover advantage. Here, innate strength creates a strategic disadvantage. Similar results have been found by other authors, but using a different equilibrium concept, e.g., risk dominance as in van Damme and Hurkens (1999 and 2004).[39]

Applying Theorem 1 requires only checking that the conditions of the theorem

[39]Using a different application, our companion paper shows that the stronger candidate does not lead in every case: It depends on the details of the strategic interaction.

hold. For the game here, this confirmation is straightforward, making it easier to characterize the sequential equilibrium than to compute the mixed strategy equilibrium – which occurs only when agents are *assumed* to move simultaneously. That assumption is sometimes motivated by analytic convenience, not because there is an economic reason to think that agents really do move simultaneously. In our electoral competition game, agents never move simultaneously. The strong candidate announces their policy first and the weak candidate responds immediately, conditioning their response on the leader's announcement.

# 5 Conclusion

We introduce a continuous-time setup to analyze two-player games where agents choose actions along with their timing. We focus on games with a second-mover advantage. Under a small set of intuitive conditions, we identify a unique SPNE entailing sequential moves by the players. The characteristics that determine which agent follows – a source of "patience" – are context-specific. An agent who is more patient in all dimensions follows in equilibrium. If each agent is more patient in a different dimension, the identity of the follower depends on the relative strength in the different dimensions.

We use this framework to study electoral competition where two candidates differ in their valence and in their competence (defined as the probability of implementing the announced policy in case of winning the election). The "stronger" candidate, with greater valence and competence, is more eager to enter the fray; this agent is less patient and therefore typically leads in equilibrium. The weaker agent then enjoys the second-mover advantage. The electoral value of greater valence and competence (intrinsic strength) always dominates the second-mover advantage, so the stronger candidate has a higher probability of winning the election. Nevertheless, the weaker candidate wins with positive probability.

Previous papers have shown that intrinsic strength may cause an agent to lose the second-mover advantage. These analyses sometimes rely on restrictive functional forms and complex calculations using risk dominance considerations. In our setup, the identity of the leader can be determined by checking a simple set of conditions, using subgame perfection as the only refinement.

Our framework offers a new perspective on timing games. It provides a useful tool to endogenize the order of players' moves, especially where players interact both in the timing of their move and in their action. Many strategic interactions are traditionally analyzed by imposing an order on the players' moves. Static games require that agents move simultaneously. In many of these games, including our static model of electoral competition, the unique Nash equilibrium is in mixed strategies. In some circumstances (e.g., with penalty kicks) the assumption of simultaneous moves is plausible, but in other cases (including electoral competition) there is no compelling reason to think that agents move simultaneously. The static assumption may be adopted because it appears to simplify an otherwise intractable game. Although the description of the dynamic game is more complicated than that of its static analog, the equilibrium analysis of the former might be much simpler. Mixed strategy equilibria, especially where action sets are continuous, are typically difficult to compute, making it hard to obtain comparative statics. The pure strategy SPNE in our dynamic game is trivial to obtain. Under a weak set of assumptions for a game where agents choose when to move, they never move simultaneously in equilibrium. In this case, equilibria where players randomize over their actions are a special feature of static simultaneous-move games.

A possible research agenda involves reconsidering static games, to determine when the assumption of simultaneous moves is descriptive. Where that assumption is made for reasons of (apparent) tractability rather than plausibility, the tools we provide may offer an alternative way to think about the game. Additional features that may be added to our continuous time modeling framework include asymmetric information or exogenous changes to the payoff structure. And last but not least, allowing players to choose the timing of their moves freely (in continuous time) in analog games that have traditionally been analyzed as static ones, promises new insights also when using laboratory experiments.[40]

[40] Calford and Oprea (2017) pioneered experiments based on the continuous time framework of Simon and Stinchcombe (1989). Our general framework (Section 3) drastically expands the set of games that can be analyzed by including infinite action sets, such as intervals.

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# A Appendix: Proofs

*Proof of Proposition 1.* Part i. We first confirm the description of $A$'s best response correspondence, and then turn to $B$'s best response correspondence.

Using Equations (3) and (5), when $\Delta a = 0$ the median voter strictly prefers candidate $A$ regardless of the realization of this voter's ideal point; in this case, candidate $A$ wins with certainty. Given the continuity of the model in actions, for all realizations of the median voter's ideal point, this voter strictly prefers candidate $A$ whenever $|\Delta a|$ is sufficiently small. Therefore, there is an interval with positive measure, that includes the $45^o$ line in Figure 2, such that for all policies in this interval candidate $A$ wins with certainty.

We now show that this interval is candidate $A$'s *entire* best response correspondence: no point outside this interval is a best response to $a_B$. To establish this claim, we show that for any pair of positions at which candidate $A$'s probability of winning is less than 1, the candidate does strictly better by moving its policy toward the $45^o$ line. We use the critical value $\bar{l}(a_A, a_B)$ in Equation (7) and compute the derivative

$$\frac{\partial \bar{l}(a_A, a_B)}{\partial a_A} = \frac{p_B(a_A - a_B)^2 + \Delta p(s - a_B)^2 + p_A p_B \Delta v}{2 p_A p_B (a_A - a_B)^2} > 0. \tag{24}$$

This relation implies that if $\pi_A < 1$, candidate $A$ has an incentive to move closer to candidate $B$'s position: If $a_A < a_B$, a higher $a_A$ increases $\bar{l}(a_A, a_B)$, thereby increasing $A$'s probability of winning, $\pi_A(a_A, a_B) = \frac{1}{2}(1 + \bar{l}(a_A, a_B))$. For $a_A > a_B$, a decrease in $a_A$ reduces $\bar{l}(a_A, a_B)$, which again implies a higher probability of winning, $\pi_A(a_A, a_B) = \frac{1}{2}(1 - \bar{l}(a_A, a_B))$. Therefore, outside the green area shown in Figure 2 there exists no interval or point where $A$ wins with certainty.

We now consider $B$'s best response correspondence. For the time being we ignore the constraints $-1 \leq \bar{l}(a_A, a_B) \leq 1$. The proof of Part (ii) establishes that these constraints are not binding, given the assumptions of the Proposition.

For $a_B < a_A$, $B$'s probability of winning the election (its objective) is $\frac{1}{2}(1 + \bar{l}(a_A, a_B))$ and for $a_B > a_A$, it is $\frac{1}{2}(1 - \bar{l}(a_A, a_B))$, so in both cases a necessary condition for a local extreme point is $\frac{\partial \bar{l}(a_A, a_B)}{\partial a_B} = 0$. Using Equation (7) and rearranging the derivative, this necessary condition implies

$$0 = \Delta v - \frac{(s - a_A)^2}{p_A} + \frac{(s - a_B)^2}{p_B} - \frac{2(s - a_B)(a_A - a_B)}{p_B}. \tag{25}$$

This equation is a quadratic in $a_B$ with roots

$$a_B^{1,2}(a_A) = a_A \pm \sqrt{\frac{\Delta p}{p_A}(s-a_A)^2 + p_B \Delta v}, \text{ with } a_B^1 > a_A > a_B^2. \tag{26}$$

We now confirm that both of these roots are *strict* local maxima. Again using (7), we obtain the second derivative

$$\frac{\partial^2 \bar{l}(a_A, a_B)}{\partial a_B^2} = -\frac{\Delta p (s-a_A)^2 + p_A p_B \Delta v}{p_A p_B (a_A - a_B)^3}.$$

The sign of this expression determines the curvature of $\bar{l}(a_A, a_B)$ with respect to $a_B$. The numerator on the right side is positive. At the larger root, $a_B = a_B^1(a_A) > a_A$; here the right side is positive, the extreme point minimizes $\bar{l}$ and thus maximizes $B$'s probability of winning. Similarly, at the smaller root, $a_B = a_B^2(a_A) < a_A$, so the extreme point maximizes $\bar{l}$ and thus maximizes $B$'s probability of winning.

Therefore, both of the roots in Equation (26) are strict local maxima. Candidate $B$'s best response to $a_A$ is the local maximum that gives $B$ the higher winning probability. Passing through the discontinuity point (as $a_A$ increases) of $B$'s best response correspondence, $B$ switches from a policy that is perceived by the voter as more rightist ($a_B^1(a_A) > a_A$) to one that is more leftist ($a_B^2(a_A) < a_A$). This switch enables $B$ to appeal to the voter under more extreme realizations of $l$.

To compute the point of discontinuity, we equalize $B$'s winning probability for $a_B^1$, $\frac{1}{2}(1-\bar{l}(a_A, a_B^1(a_A)))$, with $B$'s winning probability for $a_B^2$, $\frac{1}{2}(1+\bar{l}(a_A, a_B^2(a_A)))$, to obtain the following indifference condition:

$$\bar{l}(a_A, a_B^1(a_A)) + \bar{l}(a_A, a_B^2(a_A)) = 0. \tag{27}$$

Using Equations (7) and (26), this yields the location of the discontinuity point:

$$a_A^* = (1-p_B)s. \tag{28}$$

Note that because Equation (27) has a unique solution, for every $a_A \neq a_A^*$ one of the two local maxima gives $B$ a strictly higher probability of winning than the other: for every $a_A \neq a_A^*$, $B$'s best response correspondence is single-valued.

Part ii. The conclusion that $A$ wins with certainty if $B$ leads is an immediate consequence of Part i: $A$ merely has to match $B$'s choice sufficiently closely, i.e., it chooses any action in its best response correspondence associated with $a_B$.

We now turn to the case where $A$ leads, maintaining the assumption that the constraints $-1 \leq \bar{l}(a_A, a_B) \leq 1$ are not binding; we confirm at the end of this step that Inequality (9) is necessary and sufficient for that assumption to hold.

We need to show that if $A$ leads it chooses $a_A^*$, the discontinuity point in $B$'s best response correspondence. At this point (by construction) $B$ is indifferent between $a_B^1$ and $a_B^2$. We want to show that $a_A^*$ indeed maximizes $A$'s winning probability when $A$ leads. Recall that when $a_A < a_A^*$, as the leader, $A$ becomes the leftist candidate (because $B$ chooses $a_B^1(a_A) > a_A$ as the follower), so that $\pi_A(a_A, a_B^1(a_A)) = \frac{1}{2}(1 + \bar{l}(a_A, a_B^1(a_A)))$. Computing $\bar{l}(a_A, a_B^1(a_A))$ (using Equation (7)), we obtain (after rearranging):

$$\bar{l}(a_A, a_B^1(a_A)) = \frac{a_A - (1 - p_B)s}{p_B} + \frac{1}{p_B}\sqrt{\frac{\Delta p}{p_A}(s - a_A)^2 + p_B \Delta v}. \tag{29}$$

We need to verify that $\bar{l}(a_A, a_B^1(a_A))$ is strictly increasing in $a_A$ for $a_A < a_A^*$. To this end, we compute the derivative with respect to $a_A$ to obtain:

$$\frac{d\bar{l}(a_A, a_B^1(a_A))}{da_A} = \frac{(p_A - p_B)(a_A - s) + \sqrt{p_A \Delta p (s - a_A)^2 + p_A^2 p_B \Delta v}}{p_B \sqrt{p_A \Delta p (s - a_A)^2 + p_A^2 p_B \Delta v}}.$$

The denominator on the right side of this equality is positive, so the expression is positive if and only if the numerator is positive. The numerator is always positive if $a_A \geq s$. For $a_A < s$, notice that the numerator is strictly larger than

$$(s - a_A)\left(\sqrt{p_A \Delta p} - \Delta p\right),$$

which we obtain by setting $\Delta v = 0$ (since the numerator increases in $\Delta v$). This expression is greater than zero (for $a_A < s$) if $\sqrt{p_A \Delta p} > \Delta p$. This inequality holds because $p_A > p_B$. Therefore, we conclude that $\bar{l}(a_A, a_B^1(a_A))$ is strictly increasing in $a_A$ for $a_A < a_A^*$.

To complete the proof, one can follow similar steps as above to verify that $\frac{d\bar{l}(a_A, a_B^2(a_A))}{da_A} < 0$ when $a_A > a_A^*$, i.e., when $A$ is the rightist candidate (not shown). So in this range, $\pi_A(a_A, a_B^2(a_A))$ decreases in $a_A$.

As the leader, the optimal policy for candidate $A$ is thus $a_A^*$, the point where $B$ is indifferent between choosing $a_B^1(a_A)$ or $a_B^2(a_A)$ as the follower. At this point, $B$ is indifferent between these responses: they result in an identical winning probability for $B$. Therefore, $A$ is also indifferent between these responses at $a_A = a_A^*$. Hence, conditional on leading, $a_A^*$ maximizes $A$'s payoff.

We now confirm that the constraints $-1 \leq \bar{l}(a_A, a_B) \leq 1$ are not binding. Recall that for $a_B > a_A$, $B$'s probability of winning is $\frac{1}{2}(1 - \bar{l}(a_A, a_B))$. Here, $B$ wins with positive probability if and only if $\bar{l}(a_A, a_B) < 1$. Using the definitions $a_A^* \equiv (1 - p_B)s$ and Equation (26) for $a_B^1$ in Equation (7) and simplifying, we obtain

$$\bar{l}(a_A^*, a_B^1(a_A^*)) = \frac{1}{\sqrt{p_B}} \sqrt{\frac{p_B \Delta p}{p_A} s^2 + \Delta v}. \tag{30}$$

This expression is strictly smaller than 1 if and only if

$$\Delta v < p_B \left(1 - \frac{\Delta p}{p_A} s^2\right).$$

This reproduces Inequality (9) in the main text. The right-hand side is positive for all $p_A > p_B > 0$ and $s \in [-1, 1]$, so there always exists a positive $\Delta v$ that satisfies Inequality (9). The calculations for $a_B^2$ are similar.

Finally, we confirm that even as the leader, $A$ has a higher probability of winning than $B$, despite the fact that $B$ obtains the second-mover advantage. Consider the case where, when $A$ leads with $a_A^*$, $B$ follows with $a_B = a_B^1$, so that $a_B > a_A$ in equilibrium. (The gist of the argument when $B$ chooses $a_B^2$ is the same.) Here, $B$'s winning probability is $\pi_B = \frac{1}{2}(1 - \bar{l}(a_A^*, a_B^1(a_A^*)) < 1/2$, where the inequality uses Equation (30), which implies that $\bar{l}(a_A^*, a_B^1(a_A^*)) > 0$.

To summarize, we have shown that $\bar{l}(a_A^*, a_B^1(a_A^*)) < 1$ and $\bar{l}(a_A^*, a_B^2(a_A^*)) > -1$, so when $A$ leads and chooses $a_A = a_A^*$, $B$ has positive probability of winning. Because $a_A^*$ maximizes $A$'s probability of winning, it minimizes $B$'s probability of winning. Therefore, $B$ has a strictly higher (and thus positive) probability of winning for any other choice of $a_A$. Consequently, for all $a_A$, $B$'s optimal response guarantees $B$ a positive probability of winning: $\bar{l}(a_A, a_B^1(a_A)) < 1$ and/or $\bar{l}(a_A, a_B^2(a_A)) > -1$.

Part iii. The proof of Part ii establishes that $A$ and $B$'s best response correspondences are disjoint. Thus, there exists no pure strategy simultaneous move Nash equilibrium. □

*Proof.* (Remark 1.) We confirm the first sentence of the Remark by checking that all of the inequalities used to establish Proposition 1 continue to hold when $\Delta p = 0$. We establish the second sentence in the Remark using Equation (28), evaluated at $p_B = 1$, to obtain $a_A^* = 0$. □

*Proof of Proposition 2.* Much of the proof follows from specializing the proof of Proposition 1, setting $\Delta v = 0$. We discuss only those features that are different, including the outcome when one candidate locates at point $s$.

Part i. We can use the same argument as in Proposition 1 to establish that no point other than $a_A = a_B = s$ can be a potential pure strategy Nash equilibrium: for any $a_A = a_B \neq s$, $A$ wins with probability 1, so $B$ has an incentive to deviate, whereas for $a_B \neq s$ and $a_A \neq a_B$, $A$ has an incentive to deviate (if it does not win with probability 1) and match $a_B$ (which assures that $A$ wins). Hence, in what follows, we only need to check when $a_A = a_B = s$ is an equilibrium.

For analyzing deviations from a (potential) equilibrium, the following observation is useful: If the rival locates at $s$, a candidate may choose a different location, the same location, or choose a location arbitrarily close to $s$ ("shading"); in the latter case, for $s > 0$, shading from below (by choosing $s - \epsilon$, with $\epsilon$ arbitrarily close to zero) leads to a higher winning probability than shading from above, so we only consider the former (arguments for $s < 0$ are qualitatively the same and, thus, not shown here). Considering $a_B = s$ and $a_A = s - \epsilon$, we find (using Equation (7)): $\bar{l}(s-\epsilon, s) = s - \frac{\epsilon}{2p_A}$. Since $a_A < a_B$, this leads to a winning probability for candidate $A$ of $\pi_A = \frac{1}{2}(1+\bar{l}) = \frac{1}{2}(1+s-\frac{\epsilon}{2p_A})$ (for any $\epsilon > 0$ sufficiently small). If $A$ matches (rather than shades) $B$'s action, it obtains a winning probability of $\alpha$, so that $A$ does not shade $B$'s action for any $\alpha \geq \frac{1}{2}(1+s-\frac{\epsilon}{2p_A})$. Considering $a_A = s$ and $a_B = s - \epsilon$, we find (using Equation (7)): $\bar{l}(s, s-\epsilon) = s - \frac{\epsilon}{2p_B}$. Since $a_B < a_A$, this leads to a winning probability for candidate $B$ of $\pi_B = \frac{1}{2}(1+\bar{l}) = \frac{1}{2}(1+s-\frac{\epsilon}{2p_B})$. If $B$ matches (rather than shades) $A$'s action, it obtains a winning probability of $1-\alpha$, so that $B$ does not shade $A$'s action for $1 - \alpha \geq \frac{1}{2}(1+s-\frac{\epsilon}{2p_B})$, which never holds when $s > 0$ and $\epsilon$ is sufficiently small (given our assumption that $\alpha \geq 1/2$); $B$ thus always shades when $s > 0$.

The action pair $a_A = a_B = s$ is a pure strategy Nash equilibrium if and only if neither candidate wants to defect from that position; this condition holds if and only if both of the above inequalities are satisfied, implying

$$\frac{1}{2}(1 + s - \frac{\epsilon}{2p_A}) \leq \alpha \leq \frac{1}{2}(1 - s + \frac{\epsilon}{2p_B}) \; \forall \epsilon > 0.$$

This pair of inequalities requires $\alpha = 1/2$ and $s = 0$.

Part ii. To start, we calculate the payoffs in a sequential equilibrium where candidate $A$ leads with $a_A = a_A^*$ and $B$ follows with $a_B = a_B^1(a_A^*)$ (following with $a_B^2(a_A^*)$ leads to identical payoffs). In what follows, unless otherwise stated, we only consider the case where $s \geq 0$ (arguments for the opposite case are qualitatively the same). Using Equation (7), we find (after simplifying): $\bar{l}(a_A^*, a_B^1(a_A^*)) = s\sqrt{\frac{\Delta p}{p_A}}$. Since $a_B > a_A$, we have $\pi_B = \frac{1}{2}(1 - \bar{l})$. This yields:

$$\pi_B^{follow} \equiv \frac{1}{2}\left(1 - |s|\sqrt{\frac{\Delta p}{p_A}}\right). \tag{31}$$

The absolute value takes care of the case where $s < 0$. $A$'s payoff is $\pi_A = 1 - \pi_B$.

First consider the case where $s \neq 0$, i.e., $s > 0$. If $A$ chooses $a_A = s$ (instead of $a_A^*$), which is the only potentially profitable deviation, $B$ always shades (see part (i)). $A$'s resulting winning probability is $\pi_A = \frac{1}{2}(1 - s + \frac{\epsilon}{2p_B})$, which is smaller than 1/2 for $\epsilon$ sufficiently small (given $s > 0$). Without the deviation, $A$'s winning probability is (using Equation (31)): $\pi_A = \frac{1}{2}\left(1 + |s|\sqrt{\frac{\Delta p}{p_A}}\right) > \frac{1}{2}$. Deviating to $a_A = s$ is thus not profitable for candidate $A$. (A deviation to $s$ by candidate $B$ is not profitable by construction: $a_B^1(a_A^*)$ is $B$'s best reply.) We also observe that $B$'s winning probability (see Equation (31)) is positive since $\Delta p < p_A$.

Now consider the case where $s = 0$, which yields $a_A^* = a_B^1(a_A^*) = a_B^2(a_A^*) = 0$ in a sequential equilibrium where $A$ leads. Hence, $\pi_A = \alpha$ and $\pi_B = 1 - \alpha$. This is only an equilibrium if $\alpha = 1/2$. For $\alpha > 1/2$, $B$ responds to $a_A^* = 0$ by shading (see part (i)), which leads to winning probabilities of $\pi_B = \frac{1}{2}(1 - \frac{\epsilon}{2p_B})$, and $\pi_A = \frac{1}{2}(1 + \frac{\epsilon}{2p_B})$. The latter is at least as large as 1/2. If $A$ instead leads with some $a_A \neq 0$, $B$ responds optimally, choosing $a_B^{1,2}(a_A)$, and places itself in between $s$ and $a_A$. It is easy to verify that this leads to a winning probability of $A$ that is strictly smaller than 1/2. Hence, $A$ strictly prefers to lead with $a_A^* = 0$ also in this case, and $B$'s winning probability is again positive (equal to 1/2 in the limit $\epsilon \to 0$).

Part iii. If $B$ leads with any action $a_B \neq s$, $A$ wins with certainty by choosing a sufficiently similar position. Therefore, $B$'s best choice as leader is $a_B = s$. Focusing once more on the case where $s \geq 0$ (see above), for any $\alpha \geq \frac{1}{2}(1+s)$, $A$ has no incentive to shade $B$'s action (see part (i)). Then the winning probabilities are $\pi_A = \alpha$ and $\pi_B = 1-\alpha$. For any $\alpha < \frac{1}{2}(1+s-\frac{\epsilon}{2p_A})$, there is an $\epsilon$ sufficiently small so that $A$ prefers $a_A = s-\epsilon$ over $a_A = s$. Hence, $A$ shades $B$'s action for all $\alpha < \frac{1}{2}(1+s)$ (see part (i)). Then $B$'s winning probability is $\pi_B = \frac{1}{2}(1-s+\frac{\epsilon}{2p_A})$, which is at least as large as $\frac{1}{2}(1-s)$. In the limit ($\epsilon \to 0$), we thus obtain

$$\pi_B^{lead} \equiv \begin{cases} 1-\alpha, & \text{if } \alpha \geq \frac{1+|s|}{2} \\ \frac{1-|s|}{2}, & \text{if } \frac{1}{2} \leq \alpha < \frac{1+|s|}{2}, \end{cases}$$

a lower bound for $B$'s winning probability. (The absolute value once more takes care of the case where $s < 0$.) This can be written more concisely as:

$$\pi_B^{lead} = \min\{1-\alpha, \frac{1-|s|}{2}\}. \tag{32}$$

Candidate $B$'s winning probability is thus positive, unless $\alpha = 1$.

Part iv. Comparing Equations (31) and (32), using the fact that $\sqrt{\Delta p/p_A} < 1$, we find that

$$\pi_B^{follow} = \frac{1}{2}\left(1-|s|\sqrt{\frac{\Delta p}{p_A}}\right) > \frac{1-|s|}{2} \geq \pi_B^{lead}.$$

□

We establish two lemmas before proving Theorem 1. Lemma 1 establishes right continuity of the functions $F_j(t), L_i(t)$, and Lemma 2 provides conditions over an interval, ensuring that the players' leader-follower role does not switch.

**Lemma 1.** *Suppose that Assumptions 1–3 hold. Then*

*(i) at any $t \in [0,1)$ where $a_i^L(t)$ is continuous, $F_j(t)$ is also continuous;*

*(ii) at any $t \in [0,1)$ where $a_j^{s*}(t|t, a_i^L(t))$ and $t_j^{s*}(t|t, a_i^L(t))$ are continuous, $L_i(t)$ is also continuous; and*

*(iii) $L_i(t)$ and $F_i(t)$ are right-continuous in $t$.*

*Proof of Lemma 1.* Part (i): Due to Assumption 3 and given that (by the definition of $L_i$ and $F_j$) player $i$ leads at $t$ and player $j$ plays their optimal response to $i$'s move, a discontinuity in the follower's payoff, $F_j(t)$, can only arise at $t$ if the leader's action choice $a_i^L(t)$ changes discontinuously at $t$. Without a discontinuous change in $a_i^L(t)$, the follower's behavior (timing and/or action choice) may still change discontinuously at $t$, but only if the follower is indifferent between these types of responses. Hence, a discontinuous change in the follower's response cannot lead to a discontinuity in $F_j(t)$, if $a_i^L(t)$ is continuous at $t$.

Part (ii): Similarly as in Part (i), as long as $a_j^{s*}(t|t, a_i^L(t))$ and $t_j^{s*}(t|t, a_i^L(t))$ change continuously with $t$, a discontinuity in the leader's payoff, $L_i(t)$, cannot arise due to Assumption 3. This is because, as long as the follower's response does not change discontinuously, the leader has to be indifferent between two actions at $t$, in order to be willing to change their action discontinuously at this time. Hence, although $a_i^L(t)$ may be discontinuous at some $t$, this discontinuity does not lead to a discontinuity in $L_i(t)$, provided that $a_j^{s*}(t|t, a_i^L(t))$ and $t_j^{s*}(t|t, a_i^L(t))$ are continuous at $t$.

Part (iii): Given statements (i) and (ii) in the Lemma, it follows from Assumptions 1 and 2 that $L_i(t)$ and $F_j(t)$ are right-continuous: if there is a discontinuity in the leader's or in the follower's behavior (or both) at some $t$, then the "new" behavior (to the right of the discontinuity point in the respective player's strategy) is adopted at $t$, i.e., there is a first decision node where this happens. This implies right-continuity of the payoff functions $L_i(t)$ and $F_i(t)$. This holds also in cases where both the leader's and the follower's behavior changes discontinuously at the same $t$ (so neither case (i) nor case (ii) applies).

It remains to be shown that this behavior is not in conflict with players' optimization problems (10) and (11). In other words, we need to show that the assumption of right-continuity of the players' strategies does not in itself restrict players in a way that conflicts with their respective payoff-maximizing behavior, subject to maintaining the timing of the leader's move and holding the order of players' moves fixed. To this end, note that by their definition, the functions $a_i^L(t)$ as well as $a_j^{s*}(t|t, a_i^L(t))$ and $t_j^{s*}(t|t, a_i^L(t))$, are mutually optimized, for the given order of moves and subject to player $i$ leading at time $t$. Hence, preemption by player $j$ is ruled out by assumption.

We need only show that a player looses nothing by switching to a new type of behavior at a first decision node at some $t$, instead of immediately after time $t$. (The latter case would violate right-continuity of players' strategies.) To confirm this claim, consider some interval where players' behavior changes continuously with time, following a discontinuous change in the leader's or the follower's behavior (or both) at $t$. Moving backwards in time, the same kind of behavior as inside of this interval can also be adopted at the decision node *at* time $t$, without a discontinuous change in either player's payoff (thanks to Assumption 3). Hence, although there might exist another "mutually optimized behavior" in which players behave differently at the decision node at $t$ (for example by extending their type of behavior from immediately before $t$ onto the decision node at $t$, or by selecting another type of mutually optimized behavior if such exists), it must co-exist with the mutually optimized behavior that is selected by imposing right-continuity on the players' strategies (Assumptions 1 and 2), i.e., the type of behavior shortly after $t$. Hence, there is no strictly profitable deviation at any $t$ that is ruled out by imposing Assumptions 1 and 2, conditional on maintaining the order of players' moves and player $i$ leading at $t$. □

**Lemma 2.** *Suppose that Assumptions 1–4 hold and there is an interval* $0 \leq t_l < t_h < 1$ *such that: player* $i$ *has a strict second-mover advantage at every* $t \in (t_l, t_h]$, *and* $L_j(t)$ *is strictly decreasing at every* $t \in [t_l, t_h]$. *Suppose further that there exists a unique SPNE in the subgame that starts at* $t_h$, *such that player* $j$ *leads and player* $i$ *waits at* $t_h$. *Then player* $j$ *also leads and player* $i$ *waits in the unique SPNE at all decision nodes* $t \in [t_l, t_h)$. *In particular, the identity of the leader cannot switch at any* $t$ *in the interval* $[t_l, t_h]$.

*Proof.* By Assumption 4, there are no SPNE with simultaneous moves: all equilibria involve either sequential moves or waiting. As $j$ leads at $t_h$ and $L_j(t)$ is strictly decreasing over $[t_l, t_h]$, both players waiting over the entire interval is not an equilibrium, since moving immediately is a profitable deviation for player $j$. Therefore, any SPNE involves sequential moves.

Figure 8 provides a graphical aide for the proof, which proceeds by contradiction. Suppose to the contrary of the statement in Lemma 2, that there are decision nodes $t \in [t_l, t_h)$ where $j$ does not lead. If player $i$ also waits at those nodes, then player $j$ is strictly better off by deviating and leading at $t$, because $L_j(t)$ is strictly decreasing at every $t \in [t_l, t_h]$. If player $i$ instead leads at those decision nodes, then let $t'$ be the largest $t$ (decision node) where players "switch" their roles between leading and following – see Figure 8. (Thus, $i$ waits and $j$ leads over $[t', t_h]$.) But then there exists an interval of decision nodes prior to $t'$ where waiting until $t'$ produces a strictly higher payoff for player $i$ than leading during this interval, due to their strict second-mover advantage (see Definition 1). Moreover, if $i$ deviates from the proposed strategy and waits over this interval, then $j$'s best response is to lead in this interval. Thus, the switching strategy shown in Figure 8 cannot be part of any SPNE under the stated conditions. $\square$

Figure 8 illustrates Lemma 2.[41] The red (respectively, green) segment indicates that the player plans to wait (respectively, lead) over that interval. The proof of Lemma 2 shows that the situation in Figure 8 leads to a contradiction, and thus cannot be part of any SPNE under the stated conditions.

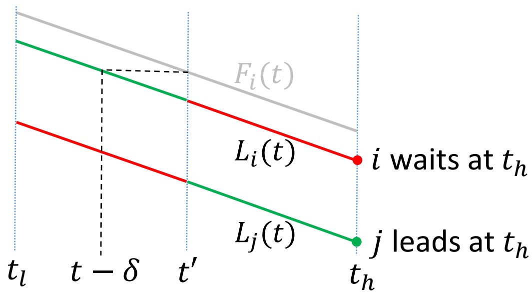


Figure 8: $L_i(t)$ green/red: player $i$ leads/waits at $t$; the case shown is *incompatible* with SPNE if player $i$ has strict second-mover advantage and $L_j(t)$ strictly decreasing

*Proof of Theorem 1.* By Assumption 4, there are no SPNE with simultaneous moves: all equilibria involve either sequential moves or waiting. We first demonstrate that the behavior described in the theorem is an equilibrium, and then that it is unique, provided that Condition 2 is satisfied.

Part i: We start with the latest subgames, and work backwards in time.

*Subames at $t \in [\tau_2, 1)$.* This step is relevant only if $\tau_2 < 1$.

We first show that both players waiting at all times $t \in [\tau_2, 1)$ is a SPNE in this part of the game. Because $\tau_2 > \tau_1$, waiting is player 1's best response if player 2 waits for $t \geq \tau_2$. For player 2, due to the definition of $\tau_i$ and by Lemma 1, it holds that $L_2(t) \leq E_2$ for all $t \geq \tau_2$. Therefore, waiting is also a best response for player 2 if player 1 waits for $t \geq \tau_2$, so that both payers waiting at all times $t \in [\tau_2, 1)$ is a SPNE.

[41]For simplicity, the figure shows $L_i$ as monotonically decreasing as well, but neither the statement of the lemma nor the proof use this feature.

We now show that this SPNE is unique, provided that Condition 2 holds. Given that there are no equilibria with simultaneous moves, we need only show that there is also no equilibrium in any subgame beginning at $t \in [\tau_2, 1)$ such that players move sequentially. We demonstrate this by contradiction. Suppose, contrary to our claim, that there exists a SPNE such that player $i$ leads at some $t \in (\tau_2, 1)$. (We treat $t = \tau_2$ separately below.) Over this interval $E_i > L_i(t)$ by the definition of $\tau_i$ and the assumption $\tau_2 > \tau_1$. Here, the only rationale for player $i$ $(i = 1, 2)$ to move at $t$ is to preempt a later move by player $j$. Thus, for such a SPNE to exist, subgames over which one player leads and the other follows would have to alternate along the time axis. If they alternated a finite number of times, there would be a final interval over which a particular player, e.g. $i$, leads and $j$ waits. However, over this interval, waiting is a profitable deviation for $i$. Therefore, in subgames over which the leader-follower roles alternate, they must do so an infinite number of times. However, that behavior contradicts Assumption 1.i, which requires that there are a finite number of discontinuity points in each player's strategy. Therefore, in the unique SPNE for all $t \in (\tau_2, 1)$, both agents wait.

*Subgame at* $t = \tau_2$: We showed above that $L_2(\tau_2) \leq E_2$. If $L_2(\tau_2) = E_2$, then player 2 is indifferent between leading or waiting at $\tau_2$. However, we have shown above that for all $t \in (\tau_2, 1)$, both agents wait. Therefore, by Assumption 1.ii, each player must also wait at $\tau_2$.

*Subgames at* $t \in [\bar{t}_2, \tau_2)$: We first show that the strategies of the two players stated in the theorem are a SPNE. Since player 1 has a strict second-mover advantage (Condition 1.i), there is no profitable deviation for player 1, who expects player 2 to move immediately at any $t$ (decision node) in this interval. There is also no profitable deviation for player 2, because player 1 is waiting and $L_2(t)$ is decreasing in this range. (Thus, a delay lowers player 2's payoff).

It remains to be shown that these strategies constitute a *unique* SPNE for subgames in this interval under Condition 2. Assumption 4 guarantees that there are no equilibria with simultaneous moves. Furthermore, there is no SPNE in which both players wait at any $t \in [\bar{t}_2, \tau_2)$, because $L_2(t) > E_2$ holds for any $t$ in this interval. Therefore, it remains to be shown that there cannot exist any sequential move SPNE other than the one stated in the theorem.

In particular, we show that there does not exist any SPNE where player 1 leads in any of these subgames. We rule out the existence of such equilibria by contradiction. Suppose, contrary to our claim, that there exists a SPNE where player 1 leads in some subgame beginning at a time $t \in (\max\{\tau_1, \bar{t}_2\}, \tau_2)$, a non-empty set. Because player 1 has a strict second-mover advantage at any $t$ in this range and $L_1(t) < E_1$, player 1's plan to lead at $t$ can be sustained in a SPNE, only if this player fears that player 2 will otherwise lead at a strictly later point in time, after a delay of at least $\delta$ units of time (for some $\delta > 0$). However, such a period of delay is incompatible with $L_2(t)$ being strictly decreasing in this range – a contradiction. It then follows that at any $t \in (\max\{\tau_1, \bar{t}_2\}, \tau_2)$, player 2 leads and player 1 waits. Thus, we can take any $t$ in this range as the "starting value" $t_h$, to initiate the procedure of working backwards in time until $\bar{t}_2$, applying Lemma 2. It follows that the proposed SPNE is unique.

Part ii The assumptions in Parts *i* and *ii* of the theorem differ only by the inclusion of Condition 1.iia to Part *ii*. That condition puts added structure only on payoffs prior to $\bar{t}_2$. It therefore does not alter (relative to Part *i*) the equilibrium for $t \in [\bar{t}_2, 1)$. The proof of Part *i* already confirms the characteristics of the equilibrium over that interval. Therefore, to verify Part *ii* of the Theorem we need only confirm that in the SPNE both players wait at subgames $t \in [0, \bar{t}_2)$. We first show that waiting over this interval is a SPNE and then we show that it is unique, given Condition 2.

By Condition 1.iia, player 1 does not benefit from preempting player 2, if player 2 plans to lead at $\bar{t}_2$, but not before. Player 2 also has no profitable deviation from the waiting strategy, because leading at $\bar{t}_2$ is more profitable than leading at any other time. Thus, the pair of waiting strategies for $t \in [0, \bar{t}_2)$ is a SPNE.

We now establish uniqueness. By Condition 1.iia, the only reason that player 1 might want to lead over this interval is to preempt its rival from leading at a later point, but prior to $\bar{t}_2$. The only reason that player 2 might want to lead at such a point is to preempt player 1 from leading at a later point, also prior to $\bar{t}_2$. Thus, a strategy involving either player leading at $t \in [0, \bar{t}_2)$ requires that there be a sequence of subgames over which the leader-follower role switches. An argument that parallels that given above for the interval $(\tau_2, 1)$ implies that this sequence must be infinite. However, an infinite sequence of switches implies that strategies are discontinuous at infinitely many points, contradicting Assumption 1. Thus, the pair of waiting strategies for $t \in [0, \bar{t}_2)$ is the unique SPNE. Given these strategies, the unique outcome of the game is for player 2 to lead at $\bar{t}_2$ and for player 1 to follow.

Part iii As with Part *ii*, the assumptions of Part *iii* differ from those of Part *i* only in placing added structure on the payoffs prior to $\bar{t}_2$. This added structure does not alter the equilibrium at subgames $t \geq \bar{t}_2$. Therefore the equilibrium for $t \geq \bar{t}_2$ described in Part *i* remains an equilibrium. We need only determine the equilibrium for $t < \bar{t}_2$.

*Subgames at* $t \in [\bar{t}_1, \bar{t}_2)$: Given that a unique SPNE has already been identified for the subgame beginning at $\bar{t}_2$, where player 2 leads, let us denote the equilibrium payoffs of the players in that subgame by $E'_2 \equiv L_2(\bar{t}_2)$, and $E'_1 \equiv F_1(\bar{t}_2)$. From the perspective of earlier subgames, these values are fixed. A move by any player before $\bar{t}_2$ effectively ends the game; the subgame at $\bar{t}_2$ can be reached only if nobody moves earlier.

Now compare the subgames starting at values of $t \in [\bar{t}_2, 1)$ with those starting at $t \in [\bar{t}_1, \bar{t}_2)$. In both cases, there is an ideal time for one agent to move, and also a latest time that they are willing to move given that the other agent follows a waiting strategy beyond that time. In between these two points $L_i(t)$ is strictly decreasing for this agent and the other agent has a strict second-mover advantage.

Therefore, the game played over $[\bar{t}_1, \bar{t}_2)$, is isomorphic to the game over the interval $[\bar{t}_2, 1)$. The roles of the two players are reversed, $\hat{t}_1$ in the former game replaces $\tau_2$ in the latter game, and $E'_i$ $(i = 1, 2)$ in the former game replaces $E_i$ in the latter.[42] The same set of assumptions applies in each of these ranges (except for the reversal of the identities of the two players). In particular, player 2 has a strict second-mover advantage in the range $[\bar{t}_1, \hat{t})$, and player 1's payoff from leading is strictly decreasing in this range. Our earlier proof for the range $[\bar{t}_2, 1)$ can thus be applied to identify a (unique) SPNE for the range $[\bar{t}_1, \bar{t}_2)$.

*Subgames at* $t \in [0, \bar{t}_1)$: In the same way, it is easy to see that the game in $[0, \bar{t}_1)$, under Condition 1.iib is isomorphic to the game in the range $[0, \bar{t}_2)$ under Condition1.iia. It thus follows from the same reasoning as in Part *ii* that the unique SPNE is for both players to wait at every $t \in [0, \bar{t}_1)$. □

*Proof of Proposition 4.* Inspection of the payoffs and the strategies confirms that Assumptions 1 – 3 of Theorem 1 are satisfied. Our parametric assumptions below equation 17 (either (i) $\Delta v > 0 \wedge \Delta p \geq 0$, or (ii) $\Delta v = 0 \wedge \Delta p > 0 \wedge s \neq 0$) together

[42]Note that, since $\bar{t}_2$ is the optimal time for player 2 to lead, there is no $t \in [0, \bar{t}_2)$ such that $L_2(t) \geq E'_2$.

with Propositions 1 and 2 imply that there is no pure strategy equilibrium in the static game. By Proposition 3 there thus exists no simultaneous move equilibria in the dynamic game either, so Assumption 4 is also satisfied.

Conditions 1.i – (iia) and 2 are also satisfied by inspection. The strict monotonicity of $L_A(t)$ required by Condition 1.i is satisfied if $\overline{\pi}_A \leq \pi_A^L$ but not if $\overline{\pi}_A > \pi_A^L$. In the latter case, Condition 4.ii enables us to use Corollary 1 to modify the equilibrium strategy from Theorem 1 for the interval $(t_A^\sharp, t_A^{crit})$. Condition 4.i ensures that $\overline{\pi}_B$ is not so large that $B$ would want to lead at the point beyond which $A$ prefers not to follow (i.e., to wait until $t = 1$). This condition implies that the definition of $\tau_B$ in this game is indeed the same as in Theorem 1. □

*Proof of Remark 3.* If Condition 4.i fails, then beginning at $t_A^{crit}$ there is a range where both candidates have a first-mover advantage. The conclusion that $B$ has a first-mover advantage follows directly from $L_B(t_A^{crit}) > 0$, and the fact that $t_A^{crit} > t_B^{crit}$ (so $B$'s payoff as follower is zero in this range). Candidate $A$ has a first-mover advantage due to our assumption $\overline{\pi}_A \geq \overline{\pi}_B$, and also $F_A(t) = 0$ because $t \geq t_A^{crit}$.

When both players have a first-mover advantage, Theorem 1 does not apply, and it is easy to see that either of the players may lead in subgames within this range. Hence, the identity of the leader at $t = t_A^{crit}$ is not uniquely determined. However, for subgames beginning at any $t$ between $t_B^{crit}$ and $t_A^{crit}$, we can uniquely identify $A$ as the leader. Shortly before $t_A^{crit}$, $A$ would rather lead than wait to enjoy their first-mover advantage. If $A$ waits until $t_A^{crit}$ and leads at that time, its payoff is lower than if it had lead earlier (because $L_A(t)$ is decreasing); if $A$ follows at $t_A^{crit}$ its payoff is zero. Thus, $A$ chooses to lead during an interval before $t_A^{crit}$. Over this interval, $B$ receives a negative payoff by leading, so $B$ waits. Reasoning backwards in time, $A$ leads at all times within this range. We can then apply the arguments of Theorem 1 for earlier subgames. □

# B Online Appendix

## B.1 Basic electoral competition game, case $\alpha < 1/2$

We show here the parts of the proof of Proposition 2 that need to be modified to include the case $\alpha < 1/2$. The results stay qualitatively the same. The modifications concern only part *ii* of the proof.

Case: $s \neq 0$ Suppose that $\alpha \leq (1-s)/2$, so that (as shown in the proof of Part (i)) $B$ has no strict incentive to shade $A$'s action (by choosing a slightly lower position). Hence, leading with $a_A = s$ gives $A$ a payoff of $\alpha$. If $A$ were to lead with any $a_A^* < a_A < s$, $B$ would pick $a_B^2(a_A) = a_A - \sqrt{\frac{\Delta p}{p_A}(s - a_A)^2}$ (see the proof of Proposition 1). $A$ would thus become the more rightist candidate. As $A$ is also the more competent candidate, a voter with $l \geq a_A$ would then always prefer candidate $A$ over $B$, and $A$'s payoff would thus be at least $(1 - a_A)/2$ which is larger than $(1-s)/2$, by $a_A < s$. If, instead, $1/2 > \alpha > (1-s)/2$, then $B$'s optimal response to $a_A = s$ is to shade that policy. As shown in Part (i), $B$ wins with probability $\frac{1}{2}(1+s-\frac{\epsilon}{2p_B})$ if it shades with $a_B = s-\epsilon$, which means $A$'s probability of winning is then $\frac{1}{2}(1-s+\frac{\epsilon}{2p_B})$. However, if $A$ would lead with $a_A^* < a_A < s-\frac{\epsilon}{2P_A}$, by the same reasoning as above, it would get a higher payoff. Hence, leading with $a_A = s$ can never be more profitable than leading with $a_A = a_A^*$.

Case: $s = 0$: If $\alpha < \frac{1}{2}$ then it is a best response for $B$ to follow with $a_B = 0$ when $A$ leads with $a_A^* = 0$. If $A$ would instead lead with $a_A = \epsilon > 0$, then $B$'s best response correspondence has this candidate follow with $a_B = \epsilon(1 - \sqrt{\frac{\Delta p}{p_A}})$. Candidate $A$ then wins with probability

$$\frac{1}{2}\left(1 - \epsilon\frac{p_B - p_A\left(1 - \sqrt{\frac{\Delta p}{p_A}}\right)^2}{2p_Ap_B\sqrt{\frac{\Delta p}{p_A}}}\right).$$

Candidate $A$'s probability of winning is thus higher if it shades 0 then if it chooses $a_A = 0$. It is easy to see that the same is true if $A$ chooses $a_A = -\epsilon$ instead.

## B.2 Basic electoral competition game, case $\Delta v < 0, \Delta p > 0$

Whereas the main text assumes that one candidate is unambiguously stronger, here we consider the case where candidate $A$ enjoys a competence advantage, while

$B$ dominates in valence.[43] Unlike the case where one candidate dominates in both characteristics, in this mixed case, the incentives to match or to differentiate away from the opponent's policy platform depend on the location in the policy space. The reason is that the competence effect $\Delta q$ changes with the distance from $s$. The game no longer has the structure of an anti-coordination game, because it is sometimes the one, and sometimes the other candidate that prefers to match or to differentiate away from the other candidate's action choice.

This is easiest to see along the diagonal where $a_A = a_B$. Here, it holds that $\Delta q|_{\Delta a=0}$ increases with the distance between $a$ and $s$ (see Equation (3)), with $\Delta q|_{\Delta a=0} = 0$ for $a = s$. Intuitively, in order to benefit from its competence advantage, candidate $A$ needs to announce a policy platform that is sufficiently different from the status quo. Otherwise, competence hardly matters. The valence advantage of candidate $B$, by contrast, is constant: it does not depend on candidates' actions. Therefore, along the diagonal, for values of $a = a_A = a_B$, $B$ wins for sure if $a$ is close to $s$; if $a$ is sufficiently far away from $s$, by contrast, $A$ wins the election.

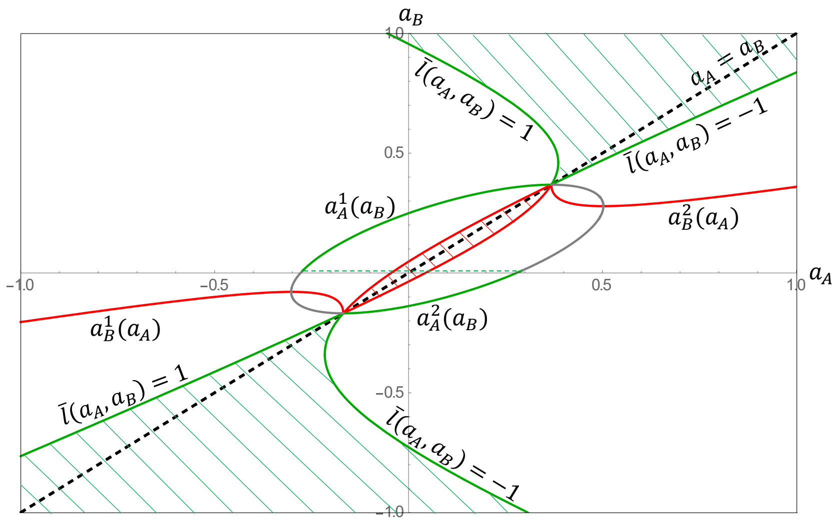


Figure 9: Best response correspondences, for $p_A = 0.9$, $p_B = 0.4$, $s = 0.1$, $\Delta v = -0.1$; candidate $A$: green; candidate $B$: red

By continuity, this logic extends to locations near the diagonal. Holding $\Delta a$ fixed at some small (positive or negative) value, when varying $a$, there is again an interval around $s$ where $B$ wins, whereas outside of this interval, $A$ wins (see Figure 9). Hence, for values of $a_A$ and $a_B$ sufficiently far away from $s$, the incentives

[43] The case where $\Delta v > 0$ and $\Delta p < 0$ is obtained by swapping the indices of the candidates.

in the game resemble those in the case where $\Delta v > 0$ and $\Delta p > 0$: $A$ prefers to match $B$'s platform, whereas $B$ prefers to differentiate away from $a_A$. By contrast, for values of $a_A$ near $s$, $B$ has an incentive to match $A$'s platform, because the valence advantage then assures that $B$ wins the election. $A$ then has an incentive to differentiate away from $a_B$ to benefit from its competence advantage. In Figure 9, the combinations of $a_A$ and $a_B$ where $B$ wins for sure (around the diagonal and near $s$) are indicated by the red striped area. The green striped areas around the diagonal further away from $s$ are the combinations for which $A$ wins for sure. For intermediate values of $a_B$, $A$'s best response is either smaller or larger than $a_B$, with a discontinuous switch between these values $(a_A^1(a_B), a_A^2(a_B))$, as illustrated by the thin dashed line (green) in Figure 9. This resembles our earlier findings (see Section 2) where $B$'s best response correspondence entailed such a switch, but now it is candidate $A$ who switches discontinuously from a rightist $(a_A > a_B)$ to a leftist policy as $a_B$ is increased. The switching point in $A$'s best response correspondence is easily calculated (using similar steps as in the proof of Proposition 1), and is located at $a_B = (1 - p_A)s$.

A formal proof is omitted, but a simple thought reveals that also in this mixed case, a pure strategy Nash equilibrium often does not exist under simultaneous moves. This is due to the constant sum property of the electoral competition game: an increase in one candidate's winning probability implies a decrease in the other's winning probability, and vice versa. Therefore, if one candidate's payoff attains a local maximum in the $a_A$-$a_B$-space, this is generally a local minimum of the other candidate's payoff function.

In Figure 9, this can be seen at the intersection points of the curves $a_B^1(a_A)$ (red) and $a_A^1(a_B)$ (grey) in the left part of the figure, as well as $a_B^2(a_A)$ (red) and $a_A^2(a_B)$ (grey) on the right. The grey color indicates that this is *not* the best reply chosen by candidate $A$ in the respective range, because a response on the other side of the political spectrum leads to a higher winning probability of this candidate.[44] Additionally, the intersection points of the curves $\bar{l}(a_A, a_B) = -1$ and $\bar{l}(a_A, a_B) = 1$, where the red striped and the green striped areas become tangent, are also not candidates for a pure strategy Nash equilibrium. At these points, at least one candidate will have an incentive to shade the policy of their rival, because this leads to a discontinuous change in candidates' winning probabilities. Excep-

[44]Results may differ when the parameter values are changed. We do not claim any generality here and content ourselves with these intuitive explanations.

tions are possible for some specific parameter constellations (see Proposition 2 and Remark 2 in Section 2).

Whether a candidate is relatively better or worse off as the leader or as the follower, depends strongly on the parameters, which can lead to very tedious case distinctions. This mixed case is thus less amenable for an extension to a game that can be analyzed in continuous time. Although this is possible, we omit this case in our dynamic analysis of the electoral competition game.

## B.3 Intuition for Theorem 1

Although the logic of the proof works "backwards in time", the discrete-time procedure of backwards induction cannot be used, because (with continuous time) between any two points in time there are infinitely many decision nodes. The procedure begins with decision nodes $t \in (\tau_2, 1)$ (see Figure 4). In this interval, neither player is willing to lead, provided that the other player also does not lead. The only reason that a player might be willing to lead would be to preempt a move by the other player. But the other player would also only lead, if this player fears that otherwise, the first player would lead. This configuration requires an infinite sequence of intervals over which players switch their roles as leader and follower, because there can be no final interval over which a player leads. That infinite sequence of switches violates Assumption 1.i, and is thus ruled out.[45] Hence, we conclude that at any decision node $t \in (\tau_2, 1)$, both players wait in any SPNE.

Building on this result, we can work backwards in time towards earlier decision nodes. At decision nodes shortly before $\tau_2$, player 1 still prefers waiting over leading, provided that player 2 does not move. But player 2 now strictly prefers leading over waiting, since $L_2(t)$ is strictly decreasing in this range. Hence, there cannot be any SPNE where both players wait at such decision nodes, or where player 1 leads. This conclusion identifies unique SPNE strategies such that player 2 leads and player 1 waits at those nodes.

This result enables us to work backwards over $[\bar{t}_2, \tau_2)$. Lemma 2 in Appendix A shows that if player $j$ leads at a point in time $t_h$ and player $i$ has a strict second-

[45]Without imposing Assumption 1.i, Theorem 1 can be maintained if another assumption is imposed or strengthened that rules out an "infinite sequence of switches". For example, if at least one player has a strict second-mover advantage also in the range $t \in (\tau_2, 1)$ (where Condition 1.i does not require it) with a strictly positive $\delta$ (see Definition 1) that applies for all $t$ in this range, the result can be restored.

mover advantage at all $t \in [t_l, t_h]$, and $L_j(t)$ is strictly declining over this interval, then players cannot switch their roles as leader and follower in this interval. This fact allows us to pin down a unique SPNE for all subgames starting at $t \in [\bar{t}_2, 1)$. As Theorem 1 states, player 2 leads at every $t \in [\bar{t}_2, \tau_2)$ whereas player 1 waits, and both players wait at every $t \in [\tau_2, 1)$.

For earlier subgames, either Condition 1.iia or 1.iib become relevant. The former rules out that player 1 would preempt player 2 before $\bar{t}_2$. This implies that both players wait at all decision nodes $t \in [0, \bar{t}_2)$, thereby completing the derivation of a unique SPNE under these conditions.

If Condition 1.iia fails, but Conditions 1.iib and 1.iii hold, the subgames in the interval $t \in [\bar{t}_1, \bar{t}_2)$ become isomorphic to those in the interval $t \in [\bar{t}_2, 1)$, with the roles of the players reversed. This fact explains why we now need to assume that player 2 (rather than player 1) has a strict second-mover advantage over part of this range. As panel B in Figure 4 illustrates, the equilibrium strategies for the first range of subgames are the mirror image of the strategies for the second range: first, the less patient player leads over a range of subgames, and then both players wait. In the outcome of the overall game, of course, player 1 will then be the leader, as these decision nodes are reached earlier. Finally, at decision nodes before $\bar{t}_1$, both players wait in the SPNE, as this is strictly more profitable for both players, given the strategy of the other player.

## B.4 Supplementary discussion (for Section 4)

If a dynamic game is based on some underlying static game, payoffs for cases where at least one player never chooses an action can usually not be inferred directly from the static game, as most static games require each player to choose an action. When extending such a game to a continuous time setting, one thus needs to clarify what happens if a player never moves. Below, we provide three examples for what could happen if a candidate in an electoral competition never announces a policy platform, and discuss what this implies for payoffs.

The simplest case is when there are only two potential candidates, and a candidate who never announces a platform loses the election with certainty, whereas the other candidate wins for sure if they announce a platform. We thus obtain $\overline{\pi}_i = \pi_i(a_i, \omega_j) = 1$ for $i = A, B$ and for all $a_i \in A_i^-$. If neither of the candidates makes a move, one may simply assume that each candidate wins with a probability of one half, or choose some other tie-breaking rule.

Our second example is where each candidate represents their party, and if a candidate never announces a policy platform, they are replaced by some other candidate.[46] Then the parameter value $\overline{\pi}_i$, capturing $i$'s payoff if $j$ never makes a move, depends on the characteristics of the new candidate by which $j$ is replaced. If the new candidate has a higher valence or higher competence than candidate $j$, this leads to $\overline{\pi}_i < \pi_i^L$. As a result, candidate $i$'s payoff function $L_i(t)$ as the leader then has a downwards discontinuity at $t_j^{crit}$, i.e., at the point in time from which $j$ would no longer enter as the follower. Conversely, if the new candidate is weaker than $j$, this leads to $\overline{\pi}_i > \pi_i^L$, amounting to an upwards discontinuity at $t_j^{crit}$. In the simplest case, $j$ is replaced by some candidate who has effectively the same characteristics as $j$, so that $\overline{\pi}_i = \pi_i^L$, and $L_i(t)$ is continuous.

In the latter case, our assumption made in the main text, that a player who leads, always leads with the same action $a_i^L$, is well justified. If candidate $j$, in case of inaction, is effectively replaced by a "clone", then as the leader, candidate $i$ finds $a_i^L$ optimal, irrespective of whether $j$ follows or remains inactive. This is because $a_i^L$ will be optimal also when playing against $j$'s clone. In other cases, one may either maintain the assumption that $i$ always chooses $a_i^L$ conditional on leading as a simplification, or work out in detail and based on a fully specified model, how the leader's optimal action changes with time, depending on whether the other candidate follows or stays out. We return to this issue below.

Our third example is a setting in which $A$ and $B$ are set as candidates, and choosing an action means to shift the policy platform away from a candidate-specific default. For example, a candidate of a leftist party may be expected to implement a leftist policy in case of winning, in line with their party's program, unless the candidate announces a different platform. Committing to a new platform may be costly to the candidate, which conforms with our assumption of a fixed cost incurred when making a move. In the following, we discuss how the parameters of the dynamic game (see Section 4) relate to the underlying electoral competition setting in this example. Small adjustments to the setting in Section 4 will be needed to accommodate this example (see below).

The values $\omega_i$ are now interpreted as specific default policies. Suppose that $\omega_i$

[46] The decision to give up candidacy is not formally captured by our framework. However, it can be anticipated from what time onwards a candidate would no longer make a move. At this point, the candidate might be replaced. See also Footnote 30 in Section 4.

is the expected policy of candidate $i$, conditional on winning the election, if this candidate never announces a policy platform.[47] We assume that by default, $B$ is the more leftist candidate ($\omega_B < \omega_A$). If candidate $i$ moves, they commit to an expected policy that (strictly) differs from $\omega_i$.[48] Adjusting the policy platform may be more effective or less costly early on, compared to later during the electoral campaign. This can explain why the payoffs are decreasing with time in this example (see Equations (16) and (17) in Section 4).

The winning probabilities if both candidates move can now be inferred directly from the underlying static game (see Section 4). For example, in the case where $\Delta v = 0$ and $\Delta p > 0$, we have $\pi_B^L = \min\{1-\alpha, \frac{1}{2}(1-|s|)\}$, as shown in Section 2 and in the proof of Proposition 2. To avoid results that depend explicitly on the value of the tie-braking parameter $\alpha$, we simply set $\alpha = 1/2$. Then $\pi_B^L = \frac{1}{2}(1-|s|)$, and $\pi_A^F = 1 - \pi_B$. If $A$ leads and $B$ follows by choosing an action, then $a_A^L = a_A^*$, and $a_B \in \{a_B^1(a_A^*), a_B^2(a_A^*)\}$, with corresponding payoffs (not shown here). As indicated in the main text, issues like entry deterrence, i.e., a strategic manipulation of the leader's action choice to prevent the other player (who would otherwise enter / move) from making a move as the follower, is not a viable strategy.[49] This is because the leader's action choice $a_i^L$ is already optimized so as to *minimize* the follower's payoff.

If nobody makes a move, each candidate maintains their respective default action $\omega_i$. Hence, we get $\pi_A = \frac{1}{2}(1 - \bar{l}(\omega_A, \omega_B))$ if $\bar{l}(\omega_A, \omega_B)$ lies in the interval $[-1, 1]$, and $\pi_B = 1 - \pi_A$. $\bar{l}(\omega_A, \omega_B)$ can be calculated using Equation (7) (not shown). Let us assume that the parameters $p_A$, $p_B$, and $s$ are such that this value indeed lies in the interval $[-1, 1]$.

To complete the description of the model, it remains to be specified what happens if only one candidate moves. The difficulty that arises in this example is that, when anticipating that the other player will no longer move, player $i$ may prefer to lead with a platform different from $a_i^L$. In particular, if the follower ($B$) does not move, as the leader, the stronger candidate $A$ has an incentive to choose

[47]Recall, that the action variable $a_i$ was defined as the expected policy, conditional on candidate $i$ winning, thereby taking into consideration $i$'s competence. In this example, candidate $i$'s competence thus matters even if $i$ never moves.

[48]In the light of candidates' fixed costs, making a move but choosing an action that coincides with $\omega_i$ or that differs only marginally from it, is not a sensible choice. Without loss of generality, we can, thus, define compact action sets that are bounded away from players' default actions.

[49]In this example, making a move is not entry, but we refer to it as "entry deterrence" here because this is a well-known concept from the literature.

a platform closer to $\omega_B$ to increase their election probability. However, in order to avoid *inducing* $j$ to move, which would be the opposite of entry deterrence, $A$ then chooses its platform under the constraint that $B$, as the follower, weakly prefers to maintain $\omega_B$. Hence, after $t_j^{crit}$, as the leader, $A$ chooses an action slightly distorted away from $a_i^L$, towards $\omega_B$, such that $B$ is indifferent between moving and not.[50]

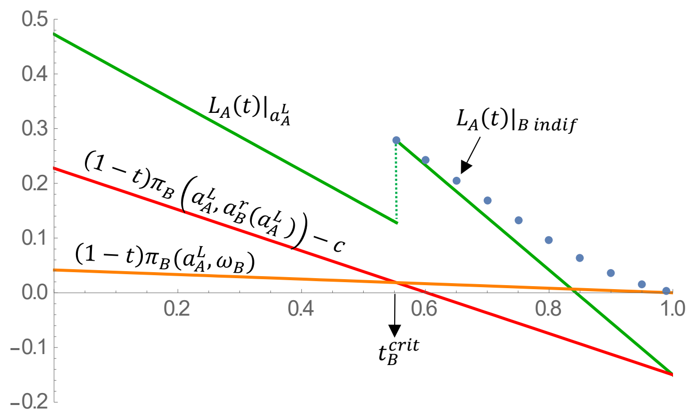


Figure 10: Dynamic electoral competition: example where candidates have default actions $\omega_A$ resp. $\omega_B$; green: $L_A(t)$, given that $A$ leads with $a_A^L$; dotted blue curve: $L_A(t)$, given that $A$ leads with action closer to $\omega_B$ such that $B$ is indifferent between moving as follower and maintaining $\omega_B$; red: $B$'s payoff as follower if $B$ moves and chooses best reply to $a_A^L$; orange: $B$'s payoff as follower if $B$ maintains $\omega_B$; for $s = 0.4$, $\Delta v = 0$, $p_A = 0.8$, $p_B = 0.5$, $c_1 = c_2 = 0.15$, $\omega_B = -0.7$

The earliest time when this issue arises is $t_B^{crit}$. Note, that $t_B^{crit}$ is now defined slightly differently than in Section 4: at this point, as the follower, $B$ is just indifferent between not making a move at all, thus maintaining $\omega_B$, and choosing their best response to the leader's action. Hence, at this point, $A$ cannot deviate from the action $a_A^L$. Because $a_A^L$ is optimized for the case where $B$ enters, leading with only a slightly different action at $t_B^{crit}$ would induce $B$ to move as the follower. Hence, at $t_B^{crit}$, the action choice $a_A^L$ remains optimal for $A$. However, at later points in time, $A$ can lead with action choices closer to $\omega_B$ while $B$ remains inactive. As a result, the curve $L_i(t)$ declines somewhat more slowly after $t_B^{crit}$, compared to the case where $B$ always leads with $a_A^L$. This is illustrated in Figure 10 (see the blue dotted curve). In the main text, we abstracted from this issue by assuming

[50]It is convenient to assume that if $B$ is indifferent, they maintain their default policy, i.e., refrain from moving.

that if candidate $i$ leads, it always leads with the same action, $a_i^L$. However, this modification does not qualitatively affect our main result (see Proposition 4).

In order to formally analyze this example, the dynamic model described in Section 4 needs to be slightly modified. In particular, in the main text, we assumed that a candidate who never announces a policy platform obtains a payoff of zero. This simplifying assumption is not well justified if choosing an action means to deviate from a default action. Then even if one player or if both players never move, each of them may still win with a positive probability. Therefore, we define additional payoffs: $\pi_i(\omega_i, a_j)$, and $\pi_i(\omega_i, \omega_j)$ for the case where player $i$ does not move, and $j$ moves resp. also does not move. In the main text, these payoffs were assumed to be zero. Calculating them is now straight-forward: because the default $\omega_i$ is an expected policy, these payoffs can be computed in the usual way by using Equation (7). Furthermore, as explained above, candidate $A$'s payoff as leader after $t_B^{crit}$ is not simply $-c_A + (1-t)\overline{\pi}_A$. Here, the parameter $\overline{\pi}_A$ used in the main text is replaced by a time-dependent payoff that needs to be computed via an indifference condition for $B$ (not shown). The resulting leader's payoff for $A$ is illustrated in Figure 10 for a numerical example. Apart from these modifications, the rest of the analysis in Section 4 remains qualitatively unchanged.[51]

## B.5 Welfare analysis

We conduct a partial welfare analysis based on the sequential SPNE identified in Section 4.[52] We content ourselves with some basic observations. We assume that the candidates and their characteristics are exogenously given. Even a social planner cannot affect candidates' valence and competence. Our goal is to analyze how (in)efficient their endogenously chosen locations in the policy space are, compared to a hypothetical case where a planner dictates these choices.[53]

[51]To apply Theorem 1 for solving the game, it must hold that the weaker candidate is unwilling to lead from an earlier time onwards than the stronger one, provided that the rival also does not lead until the end of the game ($\tau_A > \tau_B$). The modifications to $L_A(t)$ after time $t_j^{crit}$ described above (see the dotted blue curve in Figure 10) render $\tau_A$ larger while $\tau_B$ remains unchanged. Hence, the conditions needed for applying Theorem 1 are even easier to satisfy.

[52]A comprehensive welfare analysis is hard to achieve, due to the analytical complexity.

[53]We only consider the case where candidates enter sequentially at time zero, see Proposition 4. Welfare is thus the same as in the static electoral competition game (Section 2) under sequential moves, with $A$ leading.

We define welfare as follows:

$$W(a_A, a_B) = \int_{-1}^{1} U_l(w, a_w)dl, \tag{33}$$

where $w = A$ if $\Delta U_l > 0$, $w = B$ if $\Delta U_l < 0$, and $w \in \{A, B\}$ if $\Delta U_l = 0$. $U_l$ and $\Delta U_l$ are given by Equations (2) resp. (4). For given actions $a_A$ and $a_B$, the planner does not interfere with the median voter's choice of a candidate since the voter chooses optimally. Candidates' utility is neglected, including their entry costs.[54]

Note, that the definition of welfare in Equation (33) only considers the median voter, who is decisive for the outcome of the election. Hence, we neglect here the preferences of the other voters. This would necessitate additional assumptions about their "ideal policies", and how these relate to the median voter's preference. The above definition of welfare is precise if all voters share identical preferences, and it is still a good approximation if the "dispersion" of the other voters' preferences around the median voter's ideal policy is small, compared to the size of the interval over which the median voter's bliss point is distributed, i.e, $[-1, 1]$.

As a *benchmark* case, we first analyze how the planner would position a candidate, say, $A$, if only one candidate were available. Instead of the above expression for welfare, we thus maximize $\int_{-1}^{1} U_l(A, a_A)dl$. Solving this problem is straightforward and yields:

$$a_A^o = (1 - p_A)s \Leftrightarrow x_A^o = 0.$$

This result is unsurprising: if only one candidate were available, the policy platform that maximizes welfare is the center of the policy space. If both candidates are equally competent, or if $s = 0$, in a sequential equilibrium, the leader ($A$) thus positions itself in the same way as a planner would position a candidate if there were only one. The sequential equilibrium is then at least as efficient as the benchmark case where only candidate ($A$) exists, and this candidate's location is optimal.[55] In fact, the sequential equilibrium is strictly more efficient, because the weaker candidate positions itself at a different location in the policy space and is elected with a strictly positive probability (Proposition 1), so it must contribute to the (expected) welfare.[56]

[54]If candidates are citizens, their fraction of the population is negligible.

[55]Note that, due to continuity, the same welfare comparison still holds true if candidates are almost equally competent, or if $s$ is near zero.

[56]The only case where this is not true is when $\Delta v = 0$ and $s = 0$. As shown in Proposition 2, in

However, the more relevant comparison is with the case where two candidates exist, and their positions are optimized. To analyze this case, suppose $a_A > a_B$. Then Equation (33) can be rewritten as

$$W(a_A, a_B) = \int_{-1}^{\bar{l}(a_A,a_B)} U_l(B, a_B)dl + \int_{\bar{l}(a_A,a_B)}^{1} U_l(A, a_A)dl, \tag{34}$$

provided that $-1 < \bar{l}(a_A, a_B) < 1$, where $\bar{l}$ is given by Equation (7). Evaluating these integrals is straight-forward, but the resulting algebraic expressions (not shown) are too complex to lend themselves to a general analysis. We content ourselves with a simple numerical example.

Consider the following parameter values: $s = 0.2, p_A = 0.9, p_B = 0.8, v_A = 0.15, v_B = 0.1$. For a given set of parameter values, we can analyze Equation (34) numerically. We find the following optimized action choices: $a_A \approx 0.45$, and $a_B \approx -0.37$. This yields $\bar{l}(a_A, a_B) \approx -0.036$ and $W \approx -0.023$. The planner thus positions the stronger candidate $A$ to the right of the status quo policy, and the weaker one to the left of the political center. For comparison, in the sequential SPNE where the stronger candidate leads, we find: $a_A = a_A^* = 0.04$, and $a_B = a_B^2(a_A) \approx -0.17$.[57] This yields $\bar{l}(a_A, a_B) \approx -0.26$ and $W \approx -0.26$. In the benchmark case where only candidate $A$ is available, and the planner optimizes $a_A$, we find $a_A^o = 0.02$ and $W \approx -0.37$. If candidates are not too heterogeneous,[58] the existence of the second candidate thus permits to achieve a substantially higher welfare, because each candidate can then "specialize" on one part of the policy space (left or right). For the given parameter values, however, candidates' positions in the policy space are too moderate in equilibrium. The planner would force them to adopt much more extreme positions, so that for many realizations of the median voter's bliss point, one of the candidates is located nearer to the voter's ideal policy.

In the limit case where $p_A = p_B = 1$ and $\Delta v = 0$, it is easy to verify that the social optimum entails $a_A = x_A = 0.5$ and $a_B = x_B = -0.5$, with $\bar{l} = 0$.[59] The planner then specializes the candidates' positions so as to maximally "cover" the policy space. For these parameter values, our static electoral competition model collapses

that case $B$ follows by choosing $a_B = a_A$ making the voter indifferent between both candidates.

[57] Candidate $B$ is indifferent between $a_B^1(a_A)$ and $a_B^2(a_A)$, so we assume they choose the latter, which leads to a higher expected welfare.

[58] If candidates are very heterogeneous, so that one candidate clearly dominates the other one, the existence of the weaker candidate does not contribute much to the welfare optimum.

[59] Of course, due to symmetry, the indices of the candidates can be swapped.

to a standard Downsian framework without the competence or valence feature. For this setup, it is well-known that the equilibrium entails $x_A = x_B = 0$ (Median voter theorem). In our model, it is straight-forward to check that as the parameters approach these extreme values, the sequential equilibrium with $A$ leading converges to the same outcome. Both candidates then position themselves in the center of the policy space. The planner, by contrast, would force them to specialize.