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Dedicated to the memory of Andrei Aleksandrovich Gonchar and Herbert Stahl

On the analogues of Szegő's theorem for ergodic operators

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Abstract. Szegő's theorem on the asymptotic behaviour of the determinants of large Toeplitz matrices is generalized to the class of ergodic operators. The generalization is formulated in terms of a triple consisting of an ergodic operator and two functions, the symbol and the test function. It is shown that in the case of the one-dimensional discrete Schrödinger operator with random ergodic or quasiperiodic potential and various choices of the symbol and the test function this generalization leads to asymptotic formulae which have no analogues in the situation of Toeplitz operators.

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§ 1. Introduction

Szegő's theorem is an important result in analysis with a number of links and applications, see, for instance, [1]–[3] and the references there. Its original form is as follows. Let

$$\{A_j\}_{j\in\mathbb{Z}}, \qquad \overline{A_j} = A_{-j}, \quad \sum_{j\in\mathbb{Z}} |A_j| < \infty, \tag{1.1}$$

be a sequence and

$$A = \{A_{j-k}\}_{j,k\in\mathbb{Z}}, \qquad (Au)_j = \sum_{k\in\mathbb{Z}} A_{j-k} u_k, \tag{1.2}$$

be the corresponding bounded selfadjoint (discrete convolution) operator in $l^2(\mathbb{Z})$. Choose a positive integer L and consider the interval

$$\Lambda = [-L, -L+1, \dots, L] \subset \mathbb{Z} \tag{1.3}$$

and the restriction of A to Λ,

$$A_\Lambda = \{A_{j-k}\}_{j,k\in\Lambda}, \tag{1.4}$$

that is, the finite dimensional operator defined by the central $(2L+1)\times(2L+1)$ block of the doubly infinite matrix $\{A_{j-k}\}_{j,k\in\mathbb{Z}}$. Also let

$$a(p) = \sum_{j\in\mathbb{Z}} A_j e^{2\pi i p j}, \qquad p \in \mathbb{T} = [0,1), \tag{1.5}$$

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be the Fourier transform of (1.1), which is called the symbol of A in this context. Assume that

$$a(p) \geqslant 0, \qquad a \in L^1(\mathbb{T}), \tag{1.6}$$

and denote by $\{l_j\}_{j\in\mathbb{Z}}$ the Fourier coefficients of $\log a$. Then we have the two term asymptotic formula (see, for instance, [3], Theorem 1.6.2)

$$\log \operatorname{Det} A_\Lambda = |\Lambda| l_0 + \sum_{j=1}^{\infty} j l_j l_{-j} + o(1), \qquad |\Lambda| \to \infty, \tag{1.7}$$

where

$$|\Lambda| = 2L + 1. \tag{1.8}$$

The one term asymptotic formula

$$\log \operatorname{Det} A_\Lambda = |\Lambda| l_0 + o(|\Lambda|), \qquad |\Lambda| \to \infty, \tag{1.9}$$

is known as Szegő's theorem (or the first Szegő theorem), while the two term asymptotic formula (1.7) is called the strong Szegő theorem.

Note that the traditional setting for Szegő's theorem uses the Toeplitz operators defined by the semi-infinite matrix $\{A_{j-k}\}_{j,k\in\mathbb{Z}_+}$ and acting in $l^2(\mathbb{Z}_+)$. The restrictions of the Toeplitz operators are the upper left blocks $\{A_{j-k}\}_{j,k=0}^{L}$ of the semi-infinite matrix. On the other hand, in this paper we use the convolution operators (1.2) defined by the doubly infinite matrix $\{A_{j-k}\}_{j,k\in\mathbb{Z}}$, acting in $l^2(\mathbb{Z})$ and having the central blocks (1.4) as their restrictions. The latter setting seems more appropriate for our goal in this paper of dealing with ergodic operators, where the setting is standard. The same setting is widely used in multidimensional analogues of Szegő's theorem.

Using the identity $\log \operatorname{Det} A_\Lambda = \operatorname{Tr} \log A_\Lambda$ we can rewrite (1.7) as

$$\operatorname{Tr} \log A_\Lambda = |\Lambda| \int_0^1 \log a(p)\, dp + \sum_{j=1}^{\infty} j l_j l_{-j} + o(1), \qquad |\Lambda| \to \infty, \tag{1.10}$$

that is, as a two term asymptotic trace formula for the operator A_Λ, written via the 'limiting' operator A. This suggests a generalization of the formula, in which log is replaced by a function $\varphi\colon \mathbb{R} \to \mathbb{C}$ of a certain class.

The one term asymptotic formula generalizing (1.9) is well known (see, for instance, [4], § 5.2):

$$\operatorname{Tr} \varphi(A_\Lambda) = |\Lambda| \int_{\mathbb{T}} \varphi(a(p))\, dp + o(|\Lambda|), \qquad |\Lambda| \to \infty. \tag{1.11}$$

The formula is valid for any bounded continuous φ.

For the corresponding two-term asymptotic formula

$$\operatorname{Tr} \varphi(A_\Lambda) = |\Lambda| \int_{\mathbb{T}} \varphi(a(p))\, dp + T_2 + o(1), \qquad |\Lambda| \to \infty, \tag{1.12}$$

where T_2 is a functional of φ and a which is independent of Λ, see, for instance, [1], [3], [5] and the references there.

Analogous formulae hold in the continuous case, that is, for the convolution operators in $L^2(\mathbb{R})$ and their restrictions, as well as in the multidimensional case, that is, for the convolution operators in $l^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$ and their restrictions to the family $\{\Lambda_L\}$ of compact domains that expand homothetically into the whole space, see [5],

$$\Lambda_L = \left\{ x \in \mathbb{Z}^d : \frac{x}{L} \in \Lambda_1 \right\},$$

where $\Lambda_1 \in \mathbb{R}^d$ has a C^1 boundary. In this case the leading term is proportional to the volume $|\Lambda_L|$ of Λ_L, that is, to L^d, while the subleading term is proportional to L^{d-1}, that is, to the area $|\partial\Lambda_L|$ of the surface $\partial\Lambda_L$ of Λ_L, see, for instance, [1], [3] and [5]).

It is important to stress that while the leading term, proportional to L^d, of (1.12) is fairly insensitive to the smoothness of φ and a, the subleading term is proportional to L^{d-1} if φ and a are smooth enough, for instance, if a has Hölder continuous derivative and φ is C^∞ on the spectrum of A. If, however, the symbol has singularities and/or zeros, the order of the subleading term can grow more rapidly, see [2], [6] and [7]. An important example is when the symbol is the indicator of an interval Δ of the spectrum of A. In this case, if $d=1$ the two-term asymptotic formula is

$$\operatorname{Tr} \varphi(A_\Lambda) = |\Lambda|\big((1-|\Delta|)\varphi(0) + |\Delta|\varphi(1)\big) + S_2 \log|\Lambda| + o(\log|\Lambda|), \qquad |\Lambda| \to \infty, \tag{1.13}$$

where S_2 is a functional of φ, independent of $|\Lambda|$ [7]. Likewise, for $d>1$ the leading term is analogous to that for $d=1$ and, in particular, is proportional to L^d, while the subleading term is proportional to $L^{d-1}\log L$ [6]–[8].

This sensitivity of the subleading term to the smoothness of φ and a can be compared with the sensitivity of the subleading term in the Euler-Maclaurin formula for approximating the integral of a continuous function by its integral sum.

Thus, the two term asymptotic formula for $\operatorname{Tr}\varphi(A_\Lambda)$, where A_Λ is the restriction of a convolution operator A to a bounded domain, is determined by the functional parameters a and φ.

Note now that the convolution operators in $l^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$, $d \geqslant 1$ admit a natural generalization, known as ergodic (or metrically transitive) operators [9]. We recall their definition in the technically simple case of $l^2(\mathbb{Z})$.

Let

$$(\Omega, \mathscr{F}, P) \tag{1.14}$$

be a probability space and T be a measure preserving automorphism of Ω.

It is worth mentioning two important cases of the structure $(\Omega, \mathscr{F}, P, T)$ known as an abstract dynamical system. The first corresponds to the sequence $\{\xi_j\}_{j\in\mathbb{Z}}$ of i.i.d. random variables. Here $\Omega = \mathbb{R}^{\mathbb{Z}}$, $\mathscr{F}$ is the σ-algebra generated by the cylinders in $\mathbb{R}^{\mathbb{Z}}$, P is the product measure corresponding to the common probability law F of the ξ_j and

$$T\{\xi_j\}_{j\in\mathbb{Z}} = \{\xi_{j+1}\}_{j\in\mathbb{Z}} \tag{1.15}$$

is the right shift for sequences.

In the second case $\Omega = \mathbb{T}$, $\mathscr{F}$ is the Borel algebra of $\mathbb{T}$, P is the Lebesgue measure on $\mathbb{T}$ and

$$T\omega \equiv \omega + \alpha \pmod 1, \tag{1.16}$$

where $\alpha \in [0, 1)$ is an irrational number.

An important property, common to both cases, is that there is no set in $\mathscr{F}$, apart from Ω and $\varnothing$, invariant with respect to T. An abstract dynamical system possessing this property is called ergodic (or metrically transitive). The property is elementary in the second case and follows from Kolmogorov's zero-one law in the first case. From now on we will consider only ergodic dynamical systems, although certain results below are also valid without this assumption.

Now, a sequence $\{\xi_j\}_{j\in\mathbb{Z}}$ of real valued measurable functions on an abstract dynamical system is called an ergodic process if for every $t \in \mathbb{Z}$ we have

$$\xi_j(T^t\omega) = \xi_{j+t}(\omega) \quad \forall j \in \mathbb{Z} \tag{1.17}$$

with probability 1. The standard way to represent an ergodic process is via a measurable function $\mathscr{X} : \Omega \to \mathbb{R}$ and the shift operator

$$\xi_j(\omega) = \mathscr{X}(T^j\omega). \tag{1.18}$$

Likewise, a measurable function $A = \{A_{jk}\}_{j,k\in\mathbb{Z}}$ whose values are bounded operators in $l^2(\mathbb{Z})$ is called an ergodic operator if for every $t \in \mathbb{Z}$ we have

$$A_{j+t,k+t}(\omega) = A_{jk}(T^t\omega) \quad \forall j, k \in \mathbb{Z} \tag{1.19}$$

with probability 1. Choosing $\Omega = \{0\}$ in (1.14), we obtain from (1.19) that A is a convolution operator (1.2). Thus, ergodic operators comprise a generalization of convolution operators, while the latter can be viewed as nonrandom ergodic operators.

Analogous definitions can be given in the multidimensional discrete case of $\mathbb{Z}^d$ and the continuous case of $\mathbb{R}^d$, see [9], § 1D.

An important example of an ergodic operator is the one-dimensional discrete Schrödinger operator in $l^2(\mathbb{Z})$ defined as

$$H = H_0 + V, \qquad (H_0u)_j = -u_{j-1} - u_{j+1}, \quad (Vu)_j = v_ju_j, \quad j \in \mathbb{Z}, \tag{1.20}$$

and having an ergodic process $\{v_j\}_{j\in\mathbb{Z}}$ as its potential V, that is (see (1.18)),

$$v_j(\omega) = \mathscr{V}(T^j\omega), \qquad j \in \mathbb{Z}, \tag{1.21}$$

for some measurable $\mathscr{V} : \Omega \to \mathbb{R}$.

The two most widely studied cases of ergodic potentials (1.21) correspond to the two ergodic systems and processes mentioned above, that is, a sequence of i.i.d. random variables and a sequence, defined by a continuous periodic function $\mathscr{V} : \mathbb{T} \to \mathbb{R}$ and an irrational number $\alpha \in (0, 1)$ via the formula (see (1.16))

$$v_j(\omega) = \mathscr{V}(\alpha j + \omega). \tag{1.22}$$

These two classes of ergodic potentials (processes) can be viewed as the 'extreme' cases of the set of all ergodic processes with respect to the intuitive notion of 'randomness', since the i.i.d. case is intuitively the 'most random', while the quasiperiodic case is intuitively 'least random'.

An analogue of Szegő's theorems for ergodic operators could be viewed as follows. Let H be a selfadjoint ergodic operator in $l^2(\mathbb{R})$ and let $a\colon \mathbb{R} \to \mathbb{C}$ and $\varphi\colon \mathbb{C} \to \mathbb{C}$ be sufficiently 'good' functions. Then

$$A = a(H) \tag{1.23}$$

is a normal ergodic operator (see [9], Theorem 2.7). Denote its matrix by $\{A_{jk}\}_{j,k\in\mathbb{Z}}$ and the restriction of A to Λ by $A_\Lambda = \{A_{jk}\}_{j,k\in\Lambda}$ (cf. (1.4)) where Λ is given by (1.3). We are again interested in the asymptotic behaviour of the quantity

$$\operatorname{Tr}\varphi(A_\Lambda) = \operatorname{Tr}\varphi(a_\Lambda(H)), \qquad |\Lambda| \to \infty, \tag{1.24}$$

determined by the triple

$$(H, a, \varphi), \tag{1.25}$$

consisting of an underlying ergodic operator H and functions $a\colon \mathbb{R} \to \mathbb{C}$ and $\varphi\colon \mathbb{C} \to \mathbb{C}$, which we call the *symbol* and the *test function*.

In the case of discrete convolution operators corresponding to $\Omega = \{0\}$ in (1.14) the role of H is played by the selfadjoint operator $\widehat{p}$ related to the unitary shift operator

$$(U\psi)_j = \psi_{j+1}, \qquad \{\psi_j\}_{j\in\mathbb{Z}} \in l_2(\mathbb{Z}),$$

by the formula $U = e^{i\widehat{p}}$. In the case of continuous (integral) convolution operators in $L^2(\mathbb{R})$ id/dx plays the role of H. The operator $\widehat{p}$ is not widely used in this context. However, if the symbol (1.5) is even (the matrix of A in (1.2) is real and symmetric), then $a(p) = t(\cos 2\pi p)$ where

$$t(x) = \sum_{j=0}^{\infty} a_j T_j(x),$$

and $\{T_j\}$ are the Tchebyshev polynomials of the first kind: $T_j(\cos 2\pi p) = \cos 2\pi jp$. Since $\cos 2\pi p$ is the symbol of the second finite difference, that is, the Schrödinger operator H_0 of (1.20) with zero potential, we can choose this operator as H in the representation (1.23) of convolution operators. This illustrates an analogy between the Szegő's theorems for convolution and ergodic operators, according to which, in order to pass from convolution to ergodic operators, we just have to replace the Schrödinger operator H_0 with zero potential by the Schrödinger operator (1.20) with a nontrivial ergodic potential.

The leading term of (1.24) for ergodic operators is known and is a natural analogue of the leading term of (1.11)– (1.12) for convolution operators. To write down this term we need the notion of the integrated density of states of ergodic operators. Let $\{\lambda_l^{(\Lambda)}\}_{l=-L}^{L}$ be the eigenvalues (counting their multiplicity) of the restriction A_Λ of an ergodic operator A, let δ_x be the atomic measure of unit mass at $x \in \mathbb{R}$ and let

$$\mathscr{N}_\Lambda = \sum_{l=-L}^{L} \delta_{\lambda_l^{(\Lambda)}}, \qquad N_\Lambda = |\Lambda|^{-1}\mathscr{N}_\Lambda \tag{1.26}$$

be the counting measure and the normalized counting measure of eigenvalues of A_Λ. Then there exists a nonrandom nonnegative measure N of total mass 1,

$$N(\mathbb{R}) = 1, \tag{1.27}$$

such that for any piece-wise continuous bounded function $\varphi\colon \mathbb{R} \to \mathbb{C}$ we have

$$\lim_{|\Lambda|\to\infty} \int \varphi(\lambda) N_\Lambda\,(d\lambda) = \int \varphi(\lambda) N\,(d\lambda) \tag{1.28}$$

with probability 1, and for any interval $\Delta \subset \mathbb{R}$

$$N(\Delta) = \mathbf{E}\{(\mathscr{E}_A(\Delta))_{00}\}, \tag{1.29}$$

where $\mathbf{E}\{\dots\}$ denotes the expectation with respect the probability measure P of (1.14) and $\mathscr{E}_A = \{(\mathscr{E}_A)_{jk}\}_{j,k\in\mathbb{Z}}$ is the resolution of identity for A. The operator $\mathscr{E}_A(\Delta)$ is also ergodic for any $\Delta \subset \mathbb{R}$, see [9], Theorem 2.7.

The expression (1.24) can obviously be written as

$$\sum_{l=-L}^{L} \varphi(\lambda_l^{(\Lambda)}) = \int \varphi(\lambda) \mathscr{N}_\Lambda\,(d\lambda) \tag{1.30}$$

and is known in statistics as the linear statistics of the random variables $\{\lambda_l^{(\Lambda)}\}_{l=L}^{L}$ and φ is called the test function. It follows from (1.26), (1.28) and (1.29) that, with probability 1,

$$\operatorname{Tr}\varphi(A_\Lambda) = |\Lambda| \int \varphi(\lambda) N\,(d\lambda) + o(|\Lambda|) = |\Lambda| \mathbf{E}\{\varphi_{00}(A)\} + o(|\Lambda|), \qquad |\Lambda| \to \infty. \tag{1.31}$$

Note that if A is a convolution operator, then

$$(\mathscr{E}_A(\Delta))_{jk} = \int_{\mathbb{T}} e^{2\pi i(j-k)p} \chi_\Delta(a(p))\,dp, \tag{1.32}$$

where χ_Δ is the indicator function of Δ, and (1.29) implies

$$N(\Delta) = \operatorname{mes}\{p \in \mathbb{T} : a(p) \in \Delta\}.$$

As a result the right-hand side of (1.31) becomes the right-hand side of (1.11). We conclude that (1.31) can be viewed as a generalization of (1.11).

To understand the order of magnitude of the subleading term in (1.31) for ergodic operators we note first that if A is a convolution operator, then roughly speaking $\lambda_l^{(\Lambda)} = a(2\pi l/|\Lambda|)$, if Λ is large enough (see, for instance, [4], §§ 5.2–5.3). Thus, recalling the Euler-Maclaurin formula we can hope that the subleading term will be independent of Λ if a is smooth enough. This is indeed the case according to (1.7)–(1.12). On the other hand, if A is ergodic and 'sufficiently' random, then it is reasonable to suppose that its eigenvalues will also be 'sufficiently' random and then one would guess that the leading term of (1.30) has to be given by the Law of Large Numbers, that is, it has to be nonrandom and proportional to $|\Lambda|$, while the subleading term has to be given by the Central Limit Theorem, that is, it has to be $|\Lambda|^{1/2}$ times a Gaussian random variable. According to (1.31) this heuristic argument predicts the correct form of the leading term. Moreover, it was shown in [10] and [11] that the counting measure (1.26) for $\Delta = (-\infty, \mu]$, $\mu \in \mathbb{R}$, of the discrete Schrödinger operator (1.20) with an i.i.d. potential and its continuous analogue with a random Markov potential satisfies the Central Limit Theorem. This is a particular case of (1.24), in which H is the given Schrödinger operator, $a(\lambda) = \lambda$ and $\varphi(\lambda) = \chi_{(-\infty,\mu]}(\lambda)$, the indicator of $(-\infty, \mu]$, $\mu \in \mathbb{R}$. Likewise, we prove below the Central Limit Theorem for $\operatorname{Tr} \varphi(a_\Lambda(H))$, in which H is again (1.20) with random i.i.d. potential, $a(\lambda) = \lambda$ and $\varphi(\lambda) = (\lambda - x)^{-1}$ or $\varphi(\lambda) = \log(\lambda - x)$ for some $x \in \mathbb{R}$ (see Theorems 2.1 and 2.2). We conclude that the above heuristic argument applied to certain random operators correctly predicts the form of the subleading term of the two term trace formula on the left of (1.31), which can differ from that for convolution operators (that is, nonrandom ergodic operators), given by Szegő's theorem.

Moreover, in § 2 we argue that for random ergodic operators in $l^2(\mathbb{Z}^d)$, $d \geqslant 1$, the following two term asymptotic trace formula is valid in the sense of distributions:

$$\operatorname{Tr} \varphi(A_\Lambda) = L^d \int \varphi(\lambda) N\,(d\lambda) + L^{d/2}\xi + O(L^{d-1}), \qquad L \to \infty, \tag{1.33}$$

where L is a length parameter of the domain Λ (that is, $|\Lambda|$ is proportional to L^d and $|\partial\Lambda|$ is proportional to L^{d-1}), say the length of an edge of a cube centred at the origin, and ξ is the Gaussian random variable. Thus, for $d = 1$ the subleading term in (1.33) is the second term on the right, for $d = 2$ the second and the third terms are of the same order of magnitude in $|\Lambda|$, although the second term seems more important, since its Gaussian fluctuations have unbounded amplitude. Then, starting from $d = 3$ the term of the order of L^{d-1}, that is, the contribution of surface of Λ, becomes subleading (and has to be found explicitly), while the second term on the right-hand side of (1.33), which is due to the 'volume' fluctuations of the whole sum (1.30), has a lower order of magnitude than the surface term.

The above is reminiscent of the well known Larkin-Imry-Ma criterion in statistical physics on the lower critical dimension of phase transition in the ferromagnetic n-vector (or classical Heisenberg) model with short range interaction and a (quenched) random i.i.d. external field of zero mean [12], [13]. In this case the surface term $O(L^{d-1})$ is the energy of the flip of a domain wall of length L, while the energy of orientation of spins along the field is $O(L^{d/2})$. If this energy dominates, then the spins are chaotically oriented and there is no ferromagnetic order. An argument similar to the above implies that this is the case for $d = 1, 2$.

We have said that for smooth test functions and discontinuous symbols (for example, $a = \chi_\Delta$, the indicator of a spectral interval Δ) the subleading term in Szegő's asymptotic formula is logarithmic in L in the one-dimensional case (see (1.13) and [6]–[8]). On the other hand, in Theorem 2.4 we show that for $\varphi(\lambda) = \lambda(1 - \lambda)$ and $a = \chi_\Delta$, where Δ is an interval in the spectrum of the Schrödinger operator (1.20) with an i.i.d. potential, an analogue of Szegő's asymptotic formula contains neither the leading term $O(|\Lambda|)$, nor the Gaussian subleading term $O(|\Lambda|^{1/2})$ from (1.33), more precisely, it is the sum of two ergodic processes, bounded with probability 1, with respect to L in (1.3). While it is natural to expect that there will be no nonrandom 'volume' contribution $O(|\Lambda|)$ to (1.23) (in this case $a(H)$ is the spectral projection $\mathscr{E}_H(\Delta)$ and for ergodic projections the support of measure N of (1.27)–(1.29) is $\{0,1\}$, cf. the first term on the right of (1.13), the absence of the random fluctuating $O(|\Lambda|^{1/2})$ term is one more new phenomenon indicating how sensitively the term depends on the test function. Likewise, the boundedness of the 'surface term' in the case of discontinuous symbols (instead of its logarithmic form for convolution operators, see (1.13) for $d=1$) is also new and is closely related to the pure point character of the spectrum, which is quite common for one-dimensional random operators and multidimensional operators for a sufficiently large potential but is absent for the convolution operators.

We have discussed the form of the two term asymptotic trace formula in the case of random ergodic operators. There is, however, another widely studied class of both continuous and discrete ergodic operators, which we mentioned above, whose coefficients are almost periodic. Even though the stochastic properties of almost periodic coefficients are rather poor, the existence of ergodic structure proves to be rather useful in the spectral analysis of the corresponding operators. See, for instance, [14] and [9]. In § 2.2 we consider the one-dimensional Schrödinger operator (1.20) with quasiperiodic potential (1.22), where $\mathscr{V}$ is a sufficiently smooth periodic function of period 1 and $\alpha \in (0,1)$ is a Diophantine irrational number. We show (see Theorem 2.6) that for $a(\lambda)=\lambda$ and $\varphi(\lambda)=(\lambda-x)^{-1}$, where x does not lie in the spectrum of H, that is, for the same case of smooth symbol and test function as in Theorem 2.1 for the Schrödinger operator with i.i.d. potential, the two term asymptotic formula again differs from that for the convolution operators, although this time the difference is 'minimal': the leading term $O(|\Lambda|)$ of the formula is a natural analogue of the leading term of (1.12) and (1.21), however the $O(1)$ subleading term is not constant but quasiperiodic in L from (1.3). Moreover, for $\varphi(\lambda)=\lambda(1-\lambda)$ and $a=\chi_\Delta$, that is, for the same case of discontinuous symbol and smooth test function as in Theorem 2.4 for the Schrödinger operator with i.i.d. potential, the trace formula is similar to that in Theorem 2.4, provided that Δ is in the pure point spectrum of the quasiperiodic operator in question. This is again different from the logarithmic growth in $|\Lambda|$ in the corresponding trace formula for convolution operators (see more in Remark 2.5).

§ 2. The main results

In this section we present the main results of the paper and make some comments. The technical results used in the proofs are given in § 3. We stress that in what follows we will always use the ergodic Schrödinger operator (1.20) as the underlying operator H in the triple (1.25). Since the goal of the paper is to discuss new phenomena for ergodic operators in the setting of Szegő's theorem, we confine ourselves to the technically simple cases with symbols $a(\lambda)=\lambda$ and $a(\lambda)=\chi_\Delta(\lambda)$, where Δ is an interval in the spectrum of H and test functions $\varphi(\lambda)=(\lambda-x)^{-1}$, $\varphi(\lambda)=\log(\lambda-x)$, where x is outside the spectrum of H, and $\varphi(\lambda)=\lambda(1-\lambda)$. We will also comment on several other choices of the pair (a,φ). Note that the case $\varphi(\lambda)=\log(\lambda-x)$ corresponds to the original setting for Szegő's theorem, see the left-hand side of (1.10)).

We will assume for technical simplicity that the ergodic potential $\{v_j\}_{j\in\mathbb{Z}}$ in the Schrödinger operator (1.20) is bounded:

$$|v_j| \leqslant V_0 < \infty, \qquad j \in \mathbb{Z}. \tag{2.1}$$

2.1. Random operators. We will consider here the case of the intuitively 'most random' i.i.d. potentials (see the text after formula (1.22)).

Let $\sigma(H) \subset \mathbb{R}$ be the spectrum of H and let

$$G = (H - x)^{-1} = \{G_{jk}\}_{j,k\in\mathbb{Z}}, \qquad x \in \mathbb{R} \setminus \sigma(H), \tag{2.2}$$

be the resolvent of H. Denote by H_Λ the restriction of H to the interval Λ of (1.3) and by

$$G_\Lambda = (H_\Lambda - x)^{-1} = \{(G_\Lambda)_{jk}\}_{j,k\in\Lambda} \tag{2.3}$$

the resolvent of H_Λ. The bounds

$$|G_{jk}| \leqslant \|G\| \leqslant \frac{1}{\mathrm{dist}(x, \sigma(H))}, \qquad |(G_\Lambda)_{jk}| \leqslant \|G_\Lambda\| \leqslant \frac{1}{\mathrm{dist}(x, \sigma(H))}, \tag{2.4}$$

are valid for the resolvent of any selfadjoint operator. It is important that the bounds depend only on $\mathrm{dist}(x, \sigma(H))$ and, in particular, are independent of the potential and Λ.

It follows from (1.20) and (2.1) that $\|H\| \leqslant \|H_0\| + \|V\| \leqslant 2 + V_0$, hence $\sigma(H) \subset [-2 - V_0, 2 + V_0]$ and

$$\mathrm{dist}(x, \sigma(H)) \geqslant \max\{0, |x| - (2 + V_0)\}. \tag{2.5}$$

Theorem 2.1. *Let H be the Schrödinger operator (1.20) whose potential is a sequence of i.i.d. random variables satisfying (2.1) and let G and G_Λ be defined in (2.2) and (2.3). Fix $x \in \mathbb{R}$ and assume that*

$$\varepsilon := \frac{2}{|x| - V_0} \in (0, 1). \tag{2.6}$$

Then the random variable

$$|\Lambda|^{-1/2}\big(\mathrm{Tr}\, G_\Lambda - |\Lambda|\mathbf{E}\{G_{00}\}\big) \tag{2.7}$$

converges in distribution to the Gaussian random variable of zero mean and nonzero finite variance $0 < \sigma^2 < \infty$ (its form is given in (3.14)).

Proof. Note first that (2.5) and (2.6) imply that

$$\mathrm{dist}(x, \sigma(H) \geqslant \delta := \frac{2 - 2\varepsilon}{\varepsilon} > 0, \tag{2.8}$$

and, in view of (2.4), we have the bounds

$$|G_{jk}| \leqslant \|G\| \leqslant \delta^{-1} < \infty, \qquad |(G_\Lambda)_{jk}| \leqslant \|G_\Lambda\| \leqslant \delta^{-1} < \infty, \tag{2.9}$$

showing that the resolvents G and G_Λ are well defined.

Denote by $H_{\mathbb{Z}\setminus\Lambda}$ the restriction of H to the complement $\mathbb{Z}\setminus\Lambda$ of Λ. Writing the resolvent formula for the pair G and $(H_{\mathbb{Z}\setminus\Lambda}\oplus H_\Lambda - x)^{-1} = G_\Lambda(x)\oplus(H_{\mathbb{Z}\setminus\Lambda}-x)^{-1}$, we obtain

$$(G_\Lambda)_{jk} = G_{jk} + (R_\Lambda)_{jk}, \qquad j,k\in\Lambda, \tag{2.10}$$

where

$$(R_\Lambda)_{jk} = -(G_\Lambda)_{jL}G_{L+1,k} - (G_\Lambda)_{j,-L}G_{-L-1,k}. \tag{2.11}$$

Thus

$$\operatorname{Tr} G_\Lambda := \sum_{j\in\Lambda}(G_\Lambda)_{jj} = \sum_{j\in\Lambda} G_{jj} + r_\Lambda, \tag{2.12}$$

where

$$r_\Lambda = \operatorname{Tr} R_\Lambda. \tag{2.13}$$

Using the inequalities

$$\Big|\sum_{j\in\Lambda}(G_\Lambda)_{jL}G_{L+1,j}\Big|^2 \leqslant \sum_{j\in\Lambda}|(G_\Lambda)_{jL}|^2\sum_{j\in\Lambda}|G_{L+1,j}|^2 \leqslant (G_\Lambda^*G_\Lambda)_{LL}(GG^*)_{L+1,L+1},$$

analogous inequalities with $-L$ and $-(L+1)$ instead of L and $L+1$ and (2.9), we obtain

$$|r_\Lambda| \leqslant \frac{2}{\delta^2}\,, \tag{2.14}$$

that is, the second term on the right of (2.12) is $O(1)$, $|\Lambda|\to\infty$ for every realization of the random potential, since our argument is in fact deterministic.

Since the potential $V=\{v_j\}_{j\in\mathbb{Z}}$ of (1.20) is a collection of i.i.d. random variables, H is an ergodic operator (see [9], Corollary 2.6). It then follows from Theorem 2.7 and Lemma 2.8 of [9] that the resolvent $(H-z)^{-1}$ is also ergodic for $z\notin\sigma(H)$, hence $\{G_{jj}\}_{j\in\mathbb{Z}}$ is an ergodic sequence (see (1.19)), in particular,

$$\mathbf{E}\{G_{jj}\} = \mathbf{E}\{G_{00}\} \quad \forall\, j\in\mathbb{Z}.$$

This and (2.12) imply for (2.7)

$$|\Lambda|^{-1/2}\big(\operatorname{Tr} G_\Lambda - |\Lambda|\mathbf{E}\{G_{00}\}\big) = |\Lambda|^{-1/2}\sum_{j\in\Lambda}\big(G_{jj}-\mathbf{E}\{G_{jj}\}\big) + \widetilde{r}_\Lambda, \tag{2.15}$$

where

$$|\widetilde{r}_\Lambda| \leqslant \frac{4}{|\Lambda|^{1/2}\delta^2}\,. \tag{2.16}$$

Thus the proof of the theorem reduces to the proof of the Central Limit Theorem for the diagonal entries of the resolvent of the one-dimensional Schrödinger operator (1.20) with i.i.d. potential. This is proved in Lemma 3.3 under condition (2.6).

The proof is complete.

Here is one more example of the validity of the general formula (1.33) for $d=1$, an analogue of Szegő's original theorem (1.10).

Theorem 2.2. *Let H be the Schrödinger operator (1.20) whose potential is a sequence of i.i.d. random variables satisfying (2.1), let H_Λ be its restriction to Λ given in (1.3) and set*

$$L = \log(H - x) = \{L_{jk}\}_{j,k\in\mathbb{Z}}, \tag{2.17}$$
$$L_\Lambda = \log(H_\Lambda - x) = \{(L_\Lambda)_{jk}\}_{j,k\in\Lambda}, \tag{2.18}$$

where x is bounded, nonpositive and satisfies (2.6). Then the random variable

$$|\Lambda|^{-1/2}\big(\operatorname{Tr} L_\Lambda - |\Lambda|\mathbf{E}\{L_{00}\}\big)$$

converges in distribution to the Gaussian random variable of zero mean and nonzero finite variance $0 < \widehat{\sigma}^2 < \infty$ (see (3.41)).

Proof. Note first that according to (2.8) we have

$$H - x \geqslant |x| - (V_0 + 2) \geqslant \frac{2 - 2\varepsilon}{\varepsilon} > 0,$$

hence the operators (2.17) and (2.18) are well defined (cf. (1.6)).

We will use the representation

$$\log(H - x) = \log|x| + \int_0^1 H(sH - x)^{-1}\,ds \tag{2.19}$$

and its analogue for $H_\Lambda \oplus H_{\mathbb{Z}\setminus\Lambda}$. This implies (cf. (2.10)–(2.13))

$$\operatorname{Tr} L_\Lambda = \sum_{j\in\Lambda} L_{jj} + \sum_{j\in\Lambda} (\widehat{R}_\Lambda)_j,$$

where (cf. (2.10), (2.11))

$$\begin{aligned}(\widehat{R}_\Lambda)_j = &-x\int_0^1 \big((sH_\Lambda - x)^{-1}\big)_{j,L}((sH - x)^{-1})_{L+1,j}\,ds\\ &- x\int_0^1 \big((sH_\Lambda - x)^{-1}\big)_{j,-L}((sH - x)^{-1})_{-L-1,j}\,ds.\end{aligned}$$

Since the integrand in the last formula is similar to the right-hand side of (2.11), we obtain analogues of (2.15) and (2.16):

$$|\Lambda|^{-1/2}\sum_{j\in\Lambda}\big((L_\Lambda)_{jj} - \mathbf{E}\{L_{jj}\}\big) = |\Lambda|^{-1/2}\sum_{j\in\Lambda}(L_{jj} - \mathbf{E}\{G_{jj}\}) + \widehat{r}_\Lambda,$$

where

$$|\widehat{r}_\Lambda| \leqslant \frac{4x_0}{\delta^2}\,,$$

and δ is defined in (2.8).

Thus the proof of the theorem reduces to the proof of the Cental Limit Theorem for the diagonal entries of the operator L in (2.17). This is proved in Lemma 3.3.

Remark 2.3. (i) In Theorems 2.1 and 2.2 we consider triples (1.25) in which H is the one-dimensional Schrödinger operator with random potential, $a(\lambda) = \lambda$ and φ is a smooth function on the spectrum of H. In this connection it must be noted that the case with the same H and a and with $\varphi(\lambda) = \chi_{\mathbb{R}_-}(\lambda - \mu)$, $\mu \in \mathbb{R}$, that is, with a discontinuous test function, was examined in the papers [10] and [11] mentioned above. It corresponds to the linear statistic (1.30) equal to the counting function for the eigenvalues (1.26). It was shown in [10] and [11] that the random variable

$$\big(\mathscr{N}_\Lambda(\mu) - |\Lambda| N((-\infty, \mu])\big)|\Lambda|^{-1/2},$$

where $\mathscr{N}$ is defined in (1.28) and (1.29), converges in distribution to a Gaussian random variable with zero mean and positive variance, so that the Central Limit Theorem holds in this case too.

(ii) The multidimensional case $d \geqslant 1$ of Theorems 2.1 and 2.2 is similar and will be presented elsewhere. In this case we have

$$(Hu)_j = -\sum_{|k-j|=1} u_k + v_j u_j, \qquad j \in \mathbb{Z}^d,$$

instead of (1.20) and

$$(R_\Lambda)_{jk} = \sum_{|t-s|=1,\ t\in\Lambda,\ s\in\mathbb{Z}^d\setminus\Lambda} (G_\Lambda)_{jt} G_{s,k}$$

instead of (2.11). Thus, an argument similar to that leading to (2.14) yields

$$|\widetilde{r}_\Lambda| \leqslant \frac{|\partial\Lambda|}{\delta^2}$$

in this case, where now Λ is, say, a cube in $\mathbb{Z}^d$ centred at the origin and $|\partial\Lambda|$ is the number of points in $\mathbb{Z}^d$ satisfying the conditions $|t - s| = 1$, $t \in \Lambda$, $s \in \mathbb{Z}^d \setminus \Lambda$. The Central Limit Theorem for the ergodic sequence $\{G_{jj}\}_{j\in\mathbb{Z}^d}$ is also valid in this case.

Theorems 2.1 and 2.2 present two cases of the general setting (1.25) for the two term asymptotic trace formula (see (1.33)) analogous to Szegő's theorem where H is the one-dimensional Schrödinger operator with random i.i.d. potential and a and φ are smooth (even real analytic). In both cases the subleading term is given by the Central Limit Theorem. It is then reasonable to believe that the same is true for a sufficiently large class of smooth symbols and test functions. Moreover, according to [10] and [11] it is also true for certain discontinuous test functions and smooth symbols. It is also easy to show that the counting measure of the convolution operator H_0 satisfies $\mathscr{N}_\Lambda(\Delta) = c(\Delta)|\Lambda| + O(1)$ as $|\Lambda| \to \infty$. These cases demonstrate the robustness of the order of magnitude of the subleading terms in the two term asymptotic formula for (1.24) with respect to the smoothness of the test functions in the triple (1.25) (provided that the symbol is smooth) both for convolution and random ergodic operators. On the other hand, it follows from the next theorem that the asymptotic form of the trace (1.24) is sensitive to the smoothness of the symbol. Namely, we consider the triple (1.25) with $\varphi(\lambda) = \lambda(1 - \lambda)$, the same

H and $a = \chi_\Delta$, the indicator function of an interval Δ, that is, the case where $a(H) = \mathscr{E}_H(\Delta)$ is the spectral projection of H corresponding to a spectral interval Δ. This case proves to be different from both the case of convolution operators and of a discontinuous symbol (see (1.13) and [6]–[8]) and the case of random operators and smooth symbols, in particular those of [10], [11] and Theorems 2.1 and 2.2.

Theorem 2.4. *Let H be the Schrödinger operator (1.20) whose potential is a sequence of i.i.d. random variables satisfying (2.1) and having bounded probability density, let $\mathscr{E}_H(\Delta)$ be the spectral projection of H corresponding to a spectral interval $\Delta \in \sigma(H)$ such that*

$$N(\Delta) \in (0,1), \tag{2.20}$$

where N is defined in (1.27)–(1.29). Then there exist nonzero random variables $t_\pm$, measurable with respect to the σ-algebra $\mathscr{F}_{-\infty}^{\infty}$, generated by $\{v_j\}_{j\in\mathbb{Z}}$ (see (2.22), (2.26) and (2.28)) and with probability 1 giving the asymptotic formula

$$\operatorname{Tr}(\mathscr{E}_H(\Delta))_\Lambda\big(\mathbf{1}_\Lambda - (\mathscr{E}_H(\Delta))_\Lambda\big) = t_+(T^L\omega) + t_-(T^{-L}\omega) + o(1), \qquad L \to \infty, \tag{2.21}$$

for the restriction $(\mathscr{E}_H(\Delta))_\Lambda$ of $\mathscr{E}_H(\Delta)$ to the interval $\Lambda = [-L, L]$.

Proof. Denoting

$$\mathscr{E}_H(\Delta) = \{P_{jk}\}_{j,k\in\mathbb{Z}}, \tag{2.22}$$

we can write the left-hand side of (2.21) as

$$\begin{aligned}\sum_{j,k=-L}^{L} P_{jk}(\delta_{jk} - P_{jk}) &= \sum_{j=-L}^{L}\Big(P_{jj} - \sum_{k\in\mathbb{Z}}|P_{jk}|^2 + \sum_{|k|>L}|P_{jk}|^2\Big)\\ &= \sum_{j=-L}^{L}\sum_{k>L}|P_{jk}|^2 + \sum_{j=-L}^{L}\sum_{k<-L}|P_{jk}|^2,\end{aligned} \tag{2.23}$$

where we have used

$$\sum_{k\in\mathbb{Z}}|P_{jk}|^2 = P_{jj},$$

which is valid for any orthogonal projection. The first sum on the right of (2.23) is

$$t_L^+ := \sum_{j=-L}^{L}\sum_{k>L}|P_{jk}|^2 = \sum_{j=-\infty}^{L}\sum_{k>L}|P_{jk}|^2 + \sum_{j<-L}\sum_{k>L}|P_{jk}|^2 = s_L' + s_L''. \tag{2.24}$$

We will now use a basic result from the spectral analysis of the one-dimensional Schrödinger operator with an i.i.d. potential, according to which there exist $C < \infty$ and $\gamma > 0$ providing the bound

$$\mathbf{E}\{|P_{jk}(\omega)|\} \leqslant Ce^{-\gamma|j-k|}; \tag{2.25}$$

see, for instance, the recent surveys [15] and [16]. The bound is a manifestation of complete localization, that is, the pure point character of the spectrum and the exponential decay of the eigenfunctions of the one-dimensional Schrödinger operator with i.i.d. potential.

It follows from the bound and the inequality $|P_{jk}| \leqslant 1$, which is valid for any orthogonal projection, that

$$\mathbf{E}\{s''_L\} \leqslant C_1 e^{-\gamma_1 L},$$

where $C_1 < \infty$ and $\gamma_1 > 0$ do not depend on L. This and the Borel-Cantelli lemma imply that s''_L of (2.24) vanishes with probability 1 as $L \to \infty$, that is,

$$t_+^{(L)} = s'_L + o(1), \qquad L \to \infty.$$

As for the term s'_L in (2.24), recall that $\{P_{jk}(\omega)\}_{j,k\in\mathbb{Z}}$ is the matrix of the ergodic operator $\mathscr{E}_H(\Delta)$. Thus, its entries satisfy (1.19), in particular, $P_{L-j,k+L}(\omega) = P_{-j,k}(T^L\omega)$, and we can write

$$s'_L = \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} |P_{L-j,k+L}|^2 = t_+(T^L\omega),$$

where

$$t_+(\omega) = \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} |P_{-j,k}(\omega)|^2. \tag{2.26}$$

The terms in the above series are nonnegative measurable functions, thus the series is convergent with probability 1 if the series of expectations of its terms is convergent, that is, if

$$\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} \mathbf{E}\{|P_{-j,k}|^2\} < \infty.$$

This is again guaranteed by (2.25) and we conclude that $t_+(\omega)$ in (2.26) is a well-defined measurable function and that with probability 1 we have

$$t_+^{(L)} = t_+(T^L\omega)) + o(1), \qquad L \to \infty. \tag{2.27}$$

An analogous argument shows that the second sum on the right-hand side of (2.23) is $t_-(T^{-L}\omega) + o(1)$ as $L \to \infty$ with probability 1, where

$$t_-(\omega) = \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} |P_{j,-k}(\omega)|^2. \tag{2.28}$$

We will show that $t_\pm$ are not identically zero. Assume the converse, that is, say, that $t_+(\omega) = 0$ with probability 1. It then follows from (2.26) that $P_{-j,k}(\omega) = 0$ with probability 1 for $j = 0, 1, \dots$ and $k = 1, 2, \dots$. This, together with the relation $P_{s-j,k+s}(\omega) = P_{-j,k}(T^s\omega)$, which is valid for every $s \in \mathbb{Z}$ (see (1.19)), and the fact that $\mathscr{E}_H(\Delta) = \{P_{jk}\}_{j,k\in\mathbb{Z}}$ is Hermitian, imply that $P_{jk}(\omega) = 0$, $j \neq k$, that is, the spectral projection $\mathscr{E}_H(\Delta)$ is diagonal. Since $\mathscr{E}_H(\Delta)$ commutes with H, it can be either zero or the identity, so that, in view of (1.29)), $N(\Delta) \in \{0, 1\}$. However, this is incompatible with (2.20).

The same argument applies to the random variable t_- in (2.28). The proof is complete.

Remark 2.5. The argument leading to (2.23) is valid for any orthogonal projection in $l^2(\mathbb{Z})$. In addition, it follows from (1.32) for the spectral projection (2.22) of a convolution operator that

$$|P_{jk}| = \frac{|\sin \kappa(j-k)|}{|j-k|}, \qquad j,k \in \mathbb{Z}, \tag{2.29}$$

where κ is determined by a and Δ. As a result, the right-hand side of (2.23) is $\mathrm{const} \cdot \log|\Lambda|$ as $|\Lambda| \to \infty$. This is a simple example of logarithmic terms in Szegő's theorem with a discontinuous symbol, which have been mentioned more than once above (see, for instance, (1.13)).

2.2. Quasiperiodic operators. We will consider here the case where the one-dimensional discrete Schrödinger operator (1.20) with quasiperiodic potential (1.22) plays the role of H in (1.25); it is intuitively the 'least random' ergodic potential (1.18) (see the text after (1.22)).

Theorem 2.6. *Let H be the one-dimensional discrete Schrödinger operator* (1.20) *in $l^2(\mathbb{Z})$ with quasiperiodic potential* (1.22) *satisfying* (2.1). *Assume that the function $\mathscr{V}$ in* (1.22) *has $[\beta]+3$ bounded derivatives and that the frequency $\alpha \in (0,1)$ of* (1.22) *is Diophantine, that is, it admits the bound*

$$|\alpha l - m| \geqslant \frac{C}{l^{\beta}}, \qquad \beta > 1, \tag{2.30}$$

valid for all integers m and all positive integers l. Then the resolvent $G_\Lambda(\omega) = (H_\Lambda - z)^{-1}$, where H_Λ is the restriction of to Λ of (1.3),

$$\operatorname{dist}\{z, \sigma(H)\} \geqslant \eta_0 > 2, \tag{2.31}$$

and $\sigma(H)$ is the spectrum of H, satisfies:

$$\sum_{|j|\leqslant L} (G_\Lambda(\omega))_{jj} = (2L+1)\int_{\mathbb{T}} G_{00}(\omega)\,d\omega + r_+(\alpha L + \omega) + r_-(-\alpha L + \omega) + o(1), \qquad L \to \infty, \tag{2.32}$$

where $r_\pm$ are continuous 1*-periodic functions* (*see* (2.40) *for explicit formulae for $r_\pm$*).

Remark 2.7. The Diophantine numbers have Lebesgue measure 1 in $\mathbb{T}$, see [17], Theorem 32, for instance.

Proof of Theorem 2.6. From Lemma 3.5 we have

$$\begin{aligned}\sum_{|j|\leqslant L} (G_\Lambda(\omega))_{jj} &= \sum_{|j|\leqslant L} G_{jj}(\omega) - \sum_{|j|\leqslant L} G_{L,j}G_{j,L+1}(1+G_{L,L+1})^{-1} \\ &\qquad - \sum_{|j|\leqslant L} G_{-L,j}G_{j,-L-1}(1+G_{-L,-L-1})^{-1} + O(e^{-2bL}) \\ &= T_1 + T_2 + T_3 + O(e^{-2bL}), \qquad L \to \infty.\end{aligned} \tag{2.33}$$

It follows from (1.16), (1.19) and Lemma 3.1 that

$$G_{j+L,k+L}(\omega) = G_{jk}(\alpha L + \omega), \tag{2.34}$$

in particular, $G_{L,L+1}(\omega) = G_{01}(\alpha L + \omega)$. In addition, we have from (3.1) and (2.34) (cf. (2.27))

$$\begin{aligned}\sum_{j\in\Lambda} G_{L,j}(\omega)G_{j,L+1}(\omega) &= \sum_{j=-\infty}^{L} G_{L,j}(\omega)G_{j,L+1}(\omega) + O(e^{-2bL})\\ &= \sum_{j=-\infty}^{0} G_{0j}(\alpha L+\omega)G_{j1}(\alpha L+\omega) + O(e^{-2bL}), \quad L\to\infty.\end{aligned}$$

Thus the term T_2 in (2.33) is

$$\begin{aligned} T_2 &= g_+(\alpha L+\omega) + O(e^{-2bL}), \qquad L\to\infty,\\ g_+(\omega) &= (1-G_{01}(\omega))^{-1}\sum_{j=-\infty}^{0} G_{0,j}(\omega)G_{j,1}(\omega),\end{aligned} \tag{2.35}$$

that is, T_2 is quasiperiodic in L up to an exponentially small error.

The term T_3 on the right of (2.33) has an analogous asymptotic form

$$\begin{aligned} T_3 &= g_-(-\alpha L+\omega) + O(e^{-2bL}), \qquad L\to\infty,\\ g_-(\omega) &= (1-G_{0,-1}(\omega))^{-1}\sum_{j=0}^{\infty} G_{0,j}(\omega)G_{j,-1}(\omega).\end{aligned} \tag{2.36}$$

Consider now the term T_1 in (2.33). According to Lemma 3.6, G_{00} has $[\beta]+3$ continuous derivatives in $\omega\in\mathbb{T}$. Thus its Fourier coefficients

$$g_l = \int_{\mathbb{T}} e^{-2\pi i l\omega} G_{00}(\omega)\, d\omega$$

admit the bound

$$|g_l| \leqslant \frac{C}{|l|^{[\beta]+3}}, \qquad |l|\geqslant 1, \tag{2.37}$$

for some $C<\infty$. This and the Fourier series in ω for G_{00} imply that

$$\begin{aligned} T_1 := \sum_{j\in\Lambda} G_{jj}(\omega) &= (2L+1)g_0 + \sum_{l\in Z\setminus\{0\}} g_l \frac{\sin\pi\alpha l(2L+1)}{\sin\pi\alpha l} e^{2\pi i l\omega}\\ &= (2L+1)\int_{\mathbb{T}} G_{00}(\omega)\, d\omega + f_+(\alpha L+\omega) - f_-(-\alpha L+\omega),\end{aligned} \tag{2.38}$$

where

$$f_\pm(\omega) = \sum_{l\in Z\setminus\{0\}} \frac{g_l}{2i\sin\pi\alpha l} e^{2\pi i l(\omega\pm\alpha/2)}. \tag{2.39}$$

According to (2.30), for $l \neq 0$ we have

$$|\sin \pi\alpha l| = |\sin \pi(\alpha l - m)| \geqslant 2|\alpha l - m| \geqslant \frac{2C}{|l|^{\beta}},$$

and then (2.37) implies that the Fourier series on the right of (2.39) is absolutely convergent and the functions $f_{\pm}$ are continuous. This, (2.33), (2.35), (2.36) and (2.39) imply (2.32) with

$$r_{\pm}(\omega) = \pm f_{\pm}(\omega) + g_{\pm}(\omega). \tag{2.40}$$

Remark 2.8. (i) The conditions of Theorem 2.6 are not optimal. Without discussing this point in detail we just mention that the theorem is also valid for the unbounded quasiperiodic potential

$$v_j = g \tan \pi(\alpha j + \omega).$$

The corresponding proof can be obtained using the techniques presented in [19], Ch. 18.

(ii) Consider the case $\alpha = m/n$ in (1.22), where n is positive integer and $0 \leqslant m < n$, that is, the case of periodic potential of period n. Note that any periodic potential can be viewed as ergodic just by randomizing the origin of the interval of periodicity by the uniform measure. Hence, in (1.22) it suffices to consider $\omega = 0, 1/n, 2/n, \ldots, (n-1)/n$. Here, using the same argument as in the quasiperiodic case, the terms T_2 and T_3 in (2.33) are periodic in L up to an exponentially small error. To obtain the first term T_1 we write

$$|\Lambda| = 2L + 1 = \nu n + \mu n, \qquad \nu = \left[\frac{|\Lambda|}{n}\right] \in \mathbb{N}, \quad \mu = \left\{\frac{|\Lambda|}{n}\right\} \in \mathbb{Q},$$

and μ is n-periodic in $|\Lambda|$. Taking into account the periodicity of the diagonal matrix elements $\{G_{jj}\}$, for $\omega = m/n$ we obtain

$$\begin{aligned}\sum_{j\in\Lambda} G_{jj}(\omega) &= \nu \sum_{j=0}^{n-1} G_{jj}(\omega) + \sum_{j=1}^{\mu n-1} G_{jj}(\omega) \\ &= |\Lambda| n^{-1} \sum_{j=0}^{n-1} G_{00}\left(\frac{j}{n} + \omega\right) + \sum_{j=1}^{\mu n-1} G_{jj}(\omega) - \mu \sum_{j=0}^{n-1} G_{jj}(\omega),\end{aligned}$$

where the second and the third terms on the right are periodic in $|\Lambda|$ with period n, since μ has this property. To interpret this result in parallel with the quasiperiodic case we recall that $\omega = m/n$ for some $m \in [0, n-1]$ and omit ω in the first term on the right. An analogous argument applies for any periodic sequence in place of (1.22) with $\alpha = m/n$.

(iii) The argument proving the asymptotic behaviour (2.35) and (2.36) is applicable for any ergodic potential. Thus, in general the resolvent of the one-dimensional Schrödinger equation with ergodic potential (cf. (2.21)) satisfies

$$\sum_{|j|\leqslant L} (G_{\Lambda}(\omega))_{jj} = \sum_{|j|\leqslant L} G_{jj}(\omega) + g_{+}(T^{L}\omega) + g_{-}(T^{-L}\omega) + O(e^{-2bL}), \qquad L \to \infty, \tag{2.41}$$

where $g_{\pm}$ are given by (2.35) and (2.36). Furthermore, the leading term of the asymptotics for the first term on the right of (2.41) as $L \to \infty$ is again general and equals $(2L+1)\mathbf{E}\{G_{00}(\omega)\}$ by the ergodic theorem. As for the subleading term, it is either $O(L^{1/2})$ in the case of random potential (see Theorems 2.1 and 2.2 or $O(1)$ in the case of quasiperiodic potential (see (2.38)). In particular, if the potential is zero, then G is a convolution operator with the symbol $\widehat{a}(p) = (2\cos 2\pi p - z)^{-1}$. Thus, it is natural to expect that the right-hand side of (2.41) coincides with the derivative with respect to z of the right-hand side of Szegő's formula (1.10), in which $\{l_j\}_{j\in\mathbb{Z}}$ are the Fourier coefficients of $\log \widehat{a}(p)$. A standard but rather tedious calculation shows that this is indeed the case.

(iv) The main ingredient in the proof of Theorem 2.4 for the random Schrödinger operator is the bound (2.25). An analogous bound is valid for certain almost periodic operators and is closely related to the existence of the pure point component in their spectrum, typical if the potential is large enough [14], [18], [9]. Thus, the results of Theorem 2.4 are also valid for these quasiperiodic operators, which points to the rather general nature of the asymptotic formula (2.21), that is, the boundedness of the expression (1.24) for certain discontinuous symbols and Schrödinger operators with pure point spectrum. On the other hand, the absolutely continuous component of the spectrum typical for one-dimensional Schrödinger operators with quasiperiodic potentials of small amplitude is often nowhere dense (Cantor). This, with the results of [19], encourage us to believe that in this case the growth of the right-hand side of (2.23) can be quite close to linear. Note that for any orthogonal projection in $l^2(\mathbb{Z})$ the right-hand side is $o(|\Lambda|)$ as $|\Lambda| \to \infty$.

§ 3. Auxiliary results

We begin with the following lemma.

Lemma 3.1. *Let H be the one-dimensional Schrödinger operator* (1.20) *with i.i.d. random potential and G and L be defined by* (2.2) *and* (2.17). *Assume that* (2.6) *holds. Then*

(i) *there exist $C < \infty$ and $b > 0$ depending only on ε from* (2.6) *and such that*

$$|G_{jk}| \leqslant Ce^{-b|j-k|}, \qquad j,k \in \mathbb{Z}, \tag{3.1}$$

$$|L_{jk}| \leqslant Ce^{-b|j-k|}, \qquad j,k \in \mathbb{Z}, \quad x < 0; \tag{3.2}$$

(ii) *for any positive integer P we have the representation*

$$G_{jj} = G^P_{jj} + R^P_{jj}, \qquad j \in \mathbb{Z}, \tag{3.3}$$

$$L_{jj} = L^P_{jj} + \widehat{R}^P_{jj}, \qquad j \in \mathbb{Z}, \tag{3.4}$$

where G^P_{jj} and L^P_{jj} depend only on $\{v_k\}_{|j-k|\leqslant P}$ (that is, they are measurable with respect to the σ-algebra $\mathscr{F}^{j+P}_{j-P}$ generated by $\{v_k\}_{|j-k|\leqslant P}$) and

$$|G^P_{jj}| \leqslant C, \qquad |R^P_{jj}| \leqslant Ce^{-bP}, \tag{3.5}$$

$$|L^P_{jj}| \leqslant C, \qquad |\widehat{R}^P_{jj}| \leqslant Ce^{-bP}. \tag{3.6}$$

Proof. Writing out the Neumann-Liouville series for G

$$G_{jk} = \sum_{l=0}^{\infty} (-1)^l \big((V-x)^{-1}(H_0(V-x)^{-1})^l\big)_{jk} \tag{3.7}$$

and taking the inequalities $\|H_0\| \leqslant 2$ and

$$\|(V-x)^{-1}\| \leqslant \big(\mathrm{dist}\{x, [-V_0, V_0]\}\big)^{-1} \leqslant \big||x| - V_0\big|^{-1},$$

into account, we conclude that the series is convergent, since in view of (2.6),

$$\big|((V-x)^{-1}(H_0(V-x)^{-1})^l)_{jk}\big| \leqslant \big\|((V-x)^{-1}(H_0(V-x)^{-1})^l)\big\| \leqslant \varepsilon^l.$$

In addition, $(V-z)^{-1}$ is a multiplication operator and $(H_0)_{jk}$ is not zero only if $|j-k| = 1$, thus the term $(V-z)^{-1}(H_0(V-z)^{-1})^l)_{jk}$ in the series (3.7) is not zero only if $l \geqslant |j-k|$. Hence, the series starts from $l = |j-k|$ and is bounded by $\varepsilon^{|j-k|}(1-\varepsilon)^{-1}$. This is equivalent to (3.1) with

$$C = (1-\varepsilon)^{-1}, \qquad \varepsilon = e^{-b}. \tag{3.8}$$

Likewise, the first P terms of (3.7) contain $(V_t - z)^{-1}$ with $|j-t| \leqslant P$, the sum from $l = 0$ to $l = P-1$ is bounded by $(1-\varepsilon)^{-1}$ and the sum from $l = P$ to $l = \infty$ is bounded by $\varepsilon^P(1-\varepsilon)^{-1}$. Denoting the first sum by $G^P_{jj}(z)$ and the second by $R^P_{jj}(z)$ and using (3.8), we obtain (3.3) and (3.5).

The proofs of (3.2), (3.4) and (3.6) reduce to the above upon using (2.19). The proof is complete.

Remark 3.2. The bound in (3.1) is rather simple and rough. For more sophisticated bounds see [20].

Lemma 3.3. *Let the sequence $\{G_{jj}\}_{j\in\mathbb{Z}}$ be defined in* (2.2)*, where G is the resolvent of the one-dimensional Schrödinger operator* (1.20) *with random i.i.d. potential satisfying* (2.1) *and* (2.6)*. Then*:

(i) *if*

$$G^\circ_{jj} := G_{jj} - \mathbf{E}\{G_{jj}\}, \tag{3.9}$$

and

$$C_{j-k} = \mathbf{E}\{G^\circ_{jj} G^\circ_{kk}\} \tag{3.10}$$

is the covariance of $\{G_{jj}\}_{j\in\mathbb{Z}}$, then there exist $C < \infty$ and $b > 0$ such that

$$|C_{j-k}| \leqslant Ce^{-b|j-k|}; \tag{3.11}$$

(ii) *if*

$$\gamma_\Lambda = \sum_{j\in\Lambda} G_{jj}, \tag{3.12}$$

and

$$\mathbf{Var}\big\{|\Lambda|^{-1/2}\gamma_\Lambda\big\} := |\Lambda|^{-1}\mathbf{E}\big\{|\gamma_\Lambda - \mathbf{E}\{\gamma_\Lambda\}|^2\big\} \tag{3.13}$$

is its variance, then

$$\sigma^2 := \lim_{\Lambda\to\infty} \mathbf{Var}\big\{|\Lambda|^{-1/2}\gamma_\Lambda\big\} = \sum_{j\in\mathbb{Z}} C_j \in (0,\infty); \tag{3.14}$$

(iii) *the random variable*

$$|\Lambda|^{-1/2}(\gamma_\Lambda - \mathbf{E}\{\gamma_\Lambda\})$$

converges in distribution to the Gaussian random variable with zero mean and variance $\sigma^2 > 0$.

Proof. (i). The dependence of the covariance on the difference $j-k$ follows from the fact that for any $\operatorname{dist}\{x,\sigma(H)\} > 0$ (see (2.9) and (2.8)) the sequence $\{G_{jj}\}_{j\in\mathbb{Z}}$ is ergodic. Using Lemma 3.1 and (3.3)–(3.5) with $P = |j-k|/3$, we obtain that the corresponding $(G^P_{jj})^\circ$ and $(G^P_{kk})^\circ$ are independent random variables with zero mean, thus

$$\mathbf{E}\big\{(G^\circ_{jj}G^\circ_{kk}\big\} = \mathbf{E}\big\{(G^P_{jj})^\circ(R^P_{kk})^\circ\big\} + \mathbf{E}\big\{(R^P_{jj})^\circ(G^P_{kk})^\circ\big\} + \mathbf{E}\big\{(R^P_{jj})^\circ(R^P_{kk})^\circ\big\}.$$

Now, (3.3)–(3.5) imply (3.11) with $C = 12(1-\varepsilon)^{-2}$ and $e^{-b} = \varepsilon^{1/3}$.

(ii) the expression for σ^2 in (3.14) and the bound $\sigma^2 < \infty$ follows from (3.12) and (3.9)–(3.13). We will prove now that σ^2 is positive.

Note first that σ^2 is a real analytic function of x outside the spectrum of H. In addition, it is easy to establish the asymptotic formula

$$G_{jj} = -x^{-1} + v_j x^{-2} + O\bigg(\frac{1}{x^3}\bigg), \qquad |x|\to\infty,$$

and this, together with (3.9), (3.10) and (3.13), implies that

$$\sigma^2 = \mathbf{Var}\{v_0\}x^{-4}(1+O(x^{-1})), \qquad |x|\to\infty.$$

We conclude that there exists $x_1 \in (0,\infty)$ such that $\sigma^2 > 0$ if $|x| \geqslant x_1$ and σ^2 has at most a finite number of zeros if $|x| \in (x_0, x_1)$, $x_0 = 2(V_0+1)$ (see (2.6)).

We will now show that σ^2 is strictly positive for all $|x| \geqslant x_0$. Here, it is convenient to consider

$$\tau_\Lambda := \operatorname{Tr} G_\Lambda \tag{3.15}$$

instead of γ_Λ of (3.12). Indeed, from (2.12), (2.14) and (3.9)–(3.13) we have

$$|\Lambda|^{-1}\mathbf{Var}\{\gamma_\Lambda\} - |\Lambda|^{-1}\mathbf{Var}\{\tau_\Lambda\} = O(|\Lambda|^{-1/2}), \qquad |\Lambda|\to\infty. \tag{3.16}$$

To deal with $\mathbf{Var}\{\tau_\Lambda\}$ we will use a simple version of the martingale techniques (see, for instance, [21], Proposition 18.1.1), according to which if $\{X_j\}_{j=-L}^L$ are random variables, $\Phi\colon \mathbb{R}^{2L+1} \to \mathbb{R}$ is bounded and $\Psi = \Phi(X_{-L}, X_{-L+1},\dots,X_L)$, then

$$\mathbf{Var}\{\Psi\} := \mathbf{E}\big\{|\Psi - \mathbf{E}\{\Psi\}|^2\big\} = \sum_{l=-L}^{L} \mathbf{E}\big\{|\Psi^{(l)} - \Psi^{(l+1)}|^2\big\}, \tag{3.17}$$

where

$$\Psi^{(l)} = \mathbf{E}\{\Psi|\mathscr{F}_l^L\}, \qquad \Psi^{(-L)} = \Psi, \quad \Psi^{(L+1)} = \mathbf{E}\{\Psi\}, \tag{3.18}$$

and $\mathscr{F}_a^b$ is the σ-algebra generated by $\{X_j\}_{a \leqslant j \leqslant b}$.

We choose $X_j = v_j$, $|j| \leqslant L$, and $\Phi = \tau_\Lambda$, and we obtain

$$|\Lambda|^{-1}\mathbf{Var}\{\tau_\Lambda\} = |\Lambda|^{-1} \sum_{l=-L}^{L} \mathbf{E}\{|\mathscr{M}_\Lambda^{(l)}|^2\}, \qquad \mathscr{M}_\Lambda^{(l)} := \tau_\Lambda^{(l)} - \tau_\Lambda^{(l+1)}, \tag{3.19}$$

where

$$\tau_\Lambda^{(l)} = \mathbf{E}\{\tau_\Lambda|\mathscr{F}_l^L\}, \qquad \tau_\Lambda^{(-L)} = \tau_\Lambda, \quad \tau_\Lambda^{(L+1)} = \mathbf{E}\{\tau_\Lambda\}. \tag{3.20}$$

We will now make the dependence of $\tau_\Lambda^{(l)}$ on v_l explicit. To this end we use the rank one perturbation formula for the resolvents G_0 and G of a pair of selfadjoint operators A_0 and $A_0 + vP$, where $v \in \mathbb{R}$ and P is an orthogonal projection on a unit vector e:

$$G = G_0 - vG_0PG_0(1 + v(G_0e, e))^{-1}, \tag{3.21}$$

hence

$$\operatorname{Tr} G = \operatorname{Tr} G_0 - v(G_0^2e, e)(1 + v(G_0e, e))^{-1}. \tag{3.22}$$

We take $H|_{v_l=0}$ as A_0, v_l as v and the vector e_l in the canonical basis of $l^2(\mathbb{Z})$ as e. Setting

$$G_\Lambda|_{v_l=0} = \mathscr{G}_l; \tag{3.23}$$

from (3.21), (3.22) and (3.15) we obtain:

$$(G_\Lambda)_{jk} = (\mathscr{G}_l)_{jk} - v_l(\mathscr{G}_l)_{jl}(\mathscr{G}_l)_{lk}(1 + v_l(\mathscr{G}_l)_{ll})^{-1}, \tag{3.24}$$

$$\tau_\Lambda = \operatorname{Tr} \mathscr{G}_l - \frac{v_l(\mathscr{G}_l^2)_{ll}}{1 + v_l(\mathscr{G}_l)_{ll}}\,. \tag{3.25}$$

Since $\mathscr{F}_{l+1}^L$ and the σ-algebra $\mathscr{F}_l^l$ generated by v_l are independent and $\mathscr{G}_l$ does not depend on v_l, using (3.25) we have

$$\tau_\Lambda^{(l)} := \mathbf{E}\{\tau_\Lambda|\mathscr{F}_l^L\} = \mathbf{E}\{\operatorname{Tr}\mathscr{G}_l|\mathscr{F}_{l+1}^L\} - \int \mathbf{E}\left\{\frac{v_l'(\mathscr{G}_l^2)_{ll}}{1 + v_l'(\mathscr{G}_l)_{ll}}\middle|\mathscr{F}_{l+1}^L\right\}F(dv_l'),$$

$$\tau_\Lambda^{(l+1)} := \mathbf{E}\{\tau_\Lambda|\mathscr{F}_{l+1}^L\} = \mathbf{E}\{\operatorname{Tr}\mathscr{G}_l|\mathscr{F}_{l+1}^L\} - \mathbf{E}\left\{\frac{v_l(\mathscr{G}_l^2)_{ll}}{1 + v_l(\mathscr{G}_l)_{ll}}\middle|\mathscr{F}_{l+1}^L\right\},$$

where F is the common probability law of $\{v_j\}_{j\in\mathbb{Z}}$. This and (3.19) yield

$$\mathscr{M}_\Lambda^{(l)} = \mathbf{E}\left\{\frac{v_l(\mathscr{G}_l^2)_{ll}}{1 + v_l(\mathscr{G}_l)_{ll}} - \int \frac{v_l'(\mathscr{G}_l^2)_{ll}}{1 + v_l'(\mathscr{G}_l)_{ll}}F(dv_l')\middle|\mathscr{F}_{l+1}^L\right\}. \tag{3.26}$$

Since $\mathscr{G}_l$ in (3.23) does not depend on $\{v_j\}_{j\in\mathbb{Z}\setminus\Lambda}$, we can replace $\mathscr{F}_{l+1}^L$ in the above expression by $\mathscr{F}_{l+1}^\infty$. Further, recall that throughout this proof we use the finite volume resolvent G_Λ of (2.3). It follows from (2.9), (2.10) and (3.1) that we can replace G_Λ in the above formulae, in particular in (3.26), by G from (2.2) with an error that vanishes as $\Lambda \to \infty$ and this results in the relation

$$\mathscr{M}_\Lambda^{(l)} = \mathscr{M}^{(l)} + o(1), \qquad \Lambda \to \infty,$$

valid with probability 1, where

$$\mathscr{M}^{(l)} = -\mathbf{E}\left\{ \frac{v_l (G_l^2)_{ll}}{1 + v_l (G_l)_{ll}} - \int \frac{v_l' (G_l^2)_{ll}}{1 + v_l' (G_l)_{ll}} F(dv_l') \middle| \mathscr{F}_{l+1}^{\infty} \right\}, \tag{3.27}$$

$G_l = G|_{v_l=0}$, and G is the infinite-volume resolvent (2.2). This, with (3.16), allows us to write

$$\sigma^2 = \lim_{\Lambda \to \infty} |\Lambda|^{-1} \mathbf{Var}\{\gamma_\Lambda\} = \lim_{\Lambda \to \infty} |\Lambda|^{-1} \sum_{l=-L}^{L} \mathbf{E}\{|\mathscr{M}^{(l)}|^2\}. \tag{3.28}$$

Note now that the sequence $\{\mathscr{M}^{(l)}\}_{l \in \mathbb{Z}}$ of (3.27) is ergodic. A simple way to see this is to use the identity

$$\frac{v_l (G_l^2)_{ll}}{1 + v_l (G_l)_{ll}} = \frac{v_l (G^2)_{ll}}{1 - v_l (G)_{ll}} \tag{3.29}$$

(see (3.24), (3.25)), and the general relations (1.17) and (1.19). The identity follows easily from the analogue of (3.24) for the infinite volume resolvents $G_l = G|_{v_l}$ and G.

Thus, the summands in (3.28) do not depend on l and we obtain the following representation for the limiting variance:

$$\sigma^2 := \lim_{\Lambda \to \infty} |\Lambda|^{-1} \mathbf{Var}\{\gamma_\Lambda\} = \mathbf{E}\{|\mathscr{M}^{(0)}|^2\}. \tag{3.30}$$

Assume now that the right-hand side of the formula is zero. Then $\mathscr{M}^{(0)} = 0$ with probability 1, in particular, the equality

$$\mathbf{E}\left\{ \frac{v_0 (G_0^2)_{00}}{1 + v_0 (G_0)_{00}} \middle| \mathscr{F}_1^{\infty} \right\} = \mathbf{E}\left\{ \int \frac{v_0' (G_0^2)_{00}}{1 + v_0' (G_0)_{00}} F(dv_0') \middle| \mathscr{F}_1^{\infty} \right\} \tag{3.31}$$

holds for almost all v_0 in the support of F. Fixing an event from $\mathscr{F}_1^{\infty}$ and taking account of the fact that our random potential $\{v_j\}_{j \in \mathbb{Z}}$ is nontrivial we conclude that the equality is possible only if its left-hand side is independent of v_0 for fixed $(G_0)_{00}$ and $(G_0^2)_{00}$ (recall that $G_0 = G|_{v_0}$ is independent of v_0). Denoting the left-hand side of (3.31) for fixed $(G_0)_{00}$ and $(G_0^2)_{00}$ as $f(v_0)$, we obtain

$$f'(v_0) = \mathbf{E}\left\{ \frac{(G_0^2)_{00}}{(1 + v_0 (G_0)_{00})^2} \middle| \mathscr{F}_1^{\infty} \right\} = \mathbf{E}\left\{ \frac{(G^2)_{00}}{(1 - v_0 G_{00})^2} \middle| \mathscr{F}_1^{\infty} \right\}, \tag{3.32}$$

where we have used (3.29) in the last equality.

Taking (2.1), (2.9) and (2.6) into account we have

$$(1 - v_0 G_{00}) \geqslant 1 - V_0 |G_{00}| \geqslant (2 - \varepsilon)\varepsilon^{-1} > 0. \tag{3.33}$$

In addition, it follows from the spectral theorem that if $\mathscr{E} = \{\mathscr{E}_{jk}\}_{j,k \in \mathbb{Z}}$ is the resolution of the identity of H of (1.20), then we have with probability 1

$$(G^2)_{00} = \int \frac{(\mathscr{E}_H)(d\lambda))_{00}}{(\lambda - x)^2} > 0, \tag{3.34}$$

since the total mass of the measure $(\mathscr{E}_H)_{00}$ is 1 with probability 1 (one can also use (1.29)). Thus the equality $f'(v_0) = 0$ is impossible. This proves assertion (ii) of the lemma.

(iii) To prove the third assertion we use Theorem 18.6.3 from [22], according to which if $\{X_j\}_{j\in\mathbb{Z}}$ are i.i.d. random variables, $(\Omega, \mathscr{F}, P)$ is the corresponding probability space, T is the shift automorphism of Ω given by (1.15), that is, $X_{j+1} = TX_j$, $j \in \mathbb{Z}$, and Y_0 is a measurable function on the space, then the Central Limit Theorem is valid for the ergodic sequence $Y_j(\omega) = Y_0(T^j\omega)$, $j \in \mathbb{Z}$, $\omega \in \Omega$ (cf. (1.18)) if Y_0 is bounded and satisfies

$$\sum_{P=1}^{\infty} \mathbf{E}\big\{|Y_0 - \mathbf{E}\{Y_0 \mid \mathscr{F}_{-P}^{P}\}|\big\} < \infty, \tag{3.35}$$

where $\mathscr{F}_{-P}^{P}$ is the σ-algebra generated by $\{X_j\}_{|j|\leqslant P}$. We choose the i.i.d. sequence $\{v_j\}_{j\in\mathbb{Z}}$ of (1.20) as $\{X_j\}_{j\in\mathbb{Z}}$, and G_{00} with x satisfying (2.6) as Y_0, since, as G is an ergodic operator, we have $G_{jj}(\omega) = G_{00}(T^j\omega)$. To check condition (3.35) we use Lemma 3.1 with $P = [k/2]$, and obtain the bound

$$\big|G_{00} - \mathbf{E}\{G_{00}\big|\mathscr{F}_{-P}^{P}\}\big| \leqslant \big|R_{00}^{P} - \mathbf{E}\{G_{00}\big|\mathscr{F}_{-P}^{P}\}\big| \leqslant 2Ce^{-bP/2} \tag{3.36}$$

implying (3.35).

The proof of the lemma is complete.

Lemma 3.4. *Let the sequence $\{L_{jj}\}_{j\in\mathbb{Z}}$ be defined by (2.17), where H is the one-dimensional Schrödinger operator (1.20) with random i.i.d. potential satisfying the conditions of Theorem 2.2. We have:*

(i) *if*

$$L_{jj}^{\circ} := L_{jj} - \mathbf{E}\{L_{jj}\}, \tag{3.37}$$

and

$$D_{j-k} = \mathbf{E}\{L_{jj}^{\circ} L_{kk}^{\circ}\} \tag{3.38}$$

is the covariance of $\{L_{jj}\}_{j\in\mathbb{Z}}$, then there exist $C < \infty$ and $b > 0$ such that

$$|D_{j-k}| \leqslant Ce^{-b|j-k|}; \tag{3.39}$$

(ii) *if*

$$\widehat{\gamma}_\Lambda = \sum_{j\in\Lambda} L_{jj}, \tag{3.40}$$

and

$$\mathbf{Var}\big\{|\Lambda|^{-1/2}\widehat{\gamma}_\Lambda\big\} := |\Lambda|^{-1}\mathbf{E}\{|\widehat{\gamma}_\Lambda - \mathbf{E}\big\{\widehat{\gamma}_\Lambda\}|^2\big\} \tag{3.41}$$

is its variance, then

$$\widehat{\sigma}^2 := \lim_{\Lambda\to\infty} \mathbf{Var}\{|\Lambda|^{-1/2}\widehat{\gamma}_\Lambda\} = \sum_{j\in\mathbf{Z}} D_j > 0; \tag{3.42}$$

(iii) *if (2.6) holds, then the random variable*

$$|\Lambda|^{-1/2}\big(\widehat{\gamma}_\Lambda - \mathbf{E}\{\widehat{\gamma}_\Lambda\}\big)$$

converges in distribution to the Gaussian random variable with zero mean and variance $\widehat{\sigma}^2 > 0$.

Proof. We follow the scheme of proof of Lemma 3.1 and use the same notation.

(i) The proof of assertion (i) uses (2.19), (3.2), (3.4) and (3.6) instead of (3.1), (3.3) and (3.5).

(ii) Here we repeat almost word for word the argument leading to (3.20), to obtain

$$|\Lambda|^{-1}\mathbf{Var}\{\widehat{\tau}_\Lambda\} = |\Lambda|^{-1}\sum_{l=-L}^{L}\mathbf{E}\{|\widehat{\mathscr{M}}_\Lambda^{(l)}|^2\}, \qquad \widehat{\mathscr{M}}_\Lambda^{(l)} := \widehat{\tau}_\Lambda^{(l)} - \widehat{\tau}_\Lambda^{(l+1)},$$

where $\widehat{\tau}_\Lambda = \operatorname{Tr} L_\Lambda$ (cf. (3.15)) and

$$\widehat{\tau}_\Lambda^{(l)} = \mathbf{E}\{\widehat{\tau}_\Lambda|\mathscr{F}_l^L\}, \qquad \widehat{\tau}_\Lambda^{(-L)} = \widehat{\tau}_\Lambda, \quad \widehat{\tau}_\Lambda^{(L+1)} = \mathbf{E}\{\widehat{\tau}_\Lambda\}.$$

Then, instead of (3.22) we have:

$$\operatorname{Tr}\log(H_\Lambda - x) = \log\operatorname{Det}(H_\Lambda - x) = \log\operatorname{Det}\big(H_\Lambda\big|_{v_l=0} - x\big) + \log(1 + v_l(\mathscr{G}_l)_{ll}),$$

where $\mathscr{G}_l$ is defined in (3.23). This leads to an analogue of (3.3) and (3.27):

$$\widehat{\sigma}^2 := \lim_{\Lambda\to\infty}|\Lambda|^{-1}\mathbf{Var}\{\widehat{\gamma}_\Lambda\} = \mathbf{E}\{|\widehat{\mathscr{M}}^{(0)}|^2\}, \tag{3.43}$$

where (cf. (3.27) with $l = 0$)

$$\widehat{\mathscr{M}}^{(0)} = -\mathbf{E}\Big\{\log(1 + v_0(G_0)_{00}) - \int \log(1 + v_0'(G_0)_{00})F(dv_0')\big|\mathscr{F}_1^\infty\Big\}.$$

Assume now that the right-hand side of (3.43) is zero. Then $\widehat{\mathscr{M}}^{(0)} = 0$ with probability 1. This implies, by the same argument as that after formula (3.31), that the expression $\log(1 + v_0(G_0)_{00})$ is independent of v_0. On the other hand, in view of (3.22), the derivative of the expression is

$$\frac{\partial}{\partial v_0}\log(1 + v_0(G_0)_{00}) = \frac{(G_0)_{00}}{1 + v_0(G_0)_{00}} = G_{00},$$

and from the spectral theorem (cf. (3.34)) we have

$$G_{00} = \int \frac{(\mathscr{E}_H(d\lambda))_{00}}{(\lambda - x)} > 0,$$

since, according to the conditions of Theorem 2.2, x is to the left of the spectrum of H and the total mass of measure $(\mathscr{E}_H)_{00}$ is 1 with probability 1. Hence $\log(1+v_0(G_0)_{00})$ is strictly monotonic in v_0. The contradiction thus obtained proves assertion (ii) of the lemma.

(iii) To prove this assertion we again use Theorem 18.6.3 from [22], according to which we have to check condition (3.35) for $Y = L_{00}$. This is straightforward using (2.19) and (3.36).

The lemma is proved.

Lemma 3.5. *Let H be the Schrödinger operator (1.20), let H_Λ be its restriction to the interval Λ given in (1.3), and let $\{G_{jk}(z)\}_{j,k\in\mathbb{Z}}$ and $\{(G_\Lambda(z))_{jk}\}_{j,k\in\Lambda}$ be their resolvents satisfying (2.31). Then for all $j\in\Lambda$ and $b>0$ from (3.1):*

$$\begin{aligned}(G_\Lambda(z))_{jj} = G_{jj} - G_{L,j}G_{j,L+1}(1+G_{L,L+1})^{-1}\\ - G_{-L,j}G_{j,-L-1}(1+G_{-L,-L-1})^{-1} + O(e^{-2bL}), \qquad L\to\infty.\end{aligned}$$

Proof. It follows from (2.10) and (2.11) that

$$(G_\Lambda(z))_{jk} = G_{jk} - G_{j,L+1}(G_\Lambda)_{L,k} + G_{j,-L-1}(G_\Lambda)_{-L,k}, \qquad j,k\in\Lambda. \tag{3.44}$$

Writing the formula for $j=L$, we obtain

$$(G_\Lambda(z))_{Lk} = G_{L,k}(1+G_{L,L+1})^{-1} + G_{L,-L-1}(1+G_{L,L+1})^{-1}(G_\Lambda)_{-L,k}, \qquad k\in\Lambda.$$

Now relations (2.31), (3.1) and the conditions of the lemma imply that the second term on the right is $O(e^{-2bL})$, that is,

$$(G_\Lambda(z))_{L,k} = G_{L,k}(1+G_{L,L+1})^{-1} + O(e^{-2bL}), \qquad L\to\infty. \tag{3.45}$$

Analogous argument applies to the third term in (3.44), yielding (3.45) with $-L$ and $-(L+1)$ instead of L and $L+1$. Plugging these asymptotic relations into (3.44), we obtain the assertion of the lemma.

Lemma 3.6. *Under the conditions of Theorem 2.6 the matrix element G_{00} of (2.2) has $[\beta]+3$ continuous derivatives in ω.*

Proof. Denote $\mathscr{V}^{(p)}$, $p=0,1,\dots,[\beta]+3$, the multiplication operator in $l^2(\mathbb{Z})$ defined by the sequence

$$\{v_j^{(p)}\}_{j\in\mathbb{Z}}, \qquad v_j^{(p)} = \frac{\partial^p}{\partial\omega^p}\mathscr{V}(\alpha j+\omega).$$

It follows from the conditions of Theorem 2.6 that

$$\|\mathscr{V}^{(p)}\| = \sup_{j\in\mathbb{Z}}|v_j^{(p)}| \leqslant \max_{\omega\in\mathbb{T}}|\mathscr{V}(\omega)| := v^{(p)}.$$

Now, by using the resolvent identity, (2.31) and the Schwarz inequality, we obtain

$$\frac{\partial}{\partial\omega}G_{00}(\omega) = -(G\mathscr{V}^{(1)}G)_{00} \leqslant v^{(1)}\sum_{j\in\mathbb{Z}}|G_{j0}|^2 = v^{(1)}(G^2)_{00} \leqslant \frac{q^{(1)}}{\operatorname{dist}^2(z,\sigma(H))}\,.$$

An analogous argument applies to higher derivatives, which completes the proof.

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