Interacting many-particle systems in the random Kac–Luttinger model and proof of Bose–Einstein condensation

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Introduction I

- We study bosonic many-particle systems in dimension $d \geq 2$
- in a random background (Kac–Luttinger model)
- with repulsive two-body interaction between the bosons
- No temperature (or "temperature absolute zero")
- Interested whether phase transition Bose–Einstein condensation (BEC) occurs
- BEC is a macroscopic occupation of a one-particle quantum state
- To the best of our knowledge, our paper provides the first proof of BEC for systems of interacting particles in the Kac–Luttinger model, or in fact for some higher-dimensional interacting random continuum model

Introduction II: Ideal bosons in a box

- Without interaction, $T = 0$ is trivial and $T > 0$ is assumed
- $H_N = -\triangle$ on $\mathrm{L}^2(\Lambda_N)$ where $\Lambda_N = (-L_N/2, L_N/2)^d$, $d \in \mathbb{N}$
- Dirichlet b.c., thermodynamic limit: $\lim_{N\to\infty} \frac{N}{L_N^d} = \rho$ where $\rho > 0$
- Critical density ρ_c is finite for $d \geq 3$ and infinite for $d = 1, 2$
- If and only if $\rho > \rho_c$, then BEC occurs:

$$\lim_{N\to\infty} \frac{n_N^1}{N} = \frac{\rho - \rho_c}{\rho} > 0$$

- $\rightsquigarrow$ Without random potential, BEC possible only in $d \geq 3$

Introduction III: Noninteracting bosons in random potential

- $H_N^\omega = -\triangle + V^\omega$ on $\mathrm{L}^2(\Lambda_N)$ where $\Lambda_N = (-L_N/2, L_N/2)^d$, $d \in \mathbb{N}$
- $V^\omega(x) = \sum_m u(x - x_m^\omega)$ where $\{x_m^\omega\}_m$ generated by PPP on $\mathbb{R}^d$
- $u \geq 0$ or "$u = \infty$" and compactly supported
- Thermodynamic limit: $\lim_{N\to\infty} \frac{N}{L_N^d} = \rho$ where $\rho > 0$
- $\rho_c < \infty$ for every $d \geq 1$ (due to a Lifshitz tail)
- For "$u = \infty$": If and only if $\rho > \rho_c$, then BEC in probability occurs,

$$\lim_{N\to\infty} \mathbb{P}\left(\left| \frac{n_N^{1,\omega}}{N} - \frac{\rho - \rho_c}{\rho} \right| < \zeta \right) = 1 \quad \forall \zeta > 0$$

- $\rightsquigarrow$ Random potentials can trigger and enhance occurrence of BEC

Survey of Previous Results

- Noninteracting case ($T > 0$):
  - Kac, Luttinger (1973/74): $d = 3$, $u \geq 0$ or "$u = \infty$" compactly supported $\rightsquigarrow$ *Kac–Luttinger conjecture*: BEC occurs
  - Lenoble, Pastur, Zagrebnov (2004): g-BEC in random potentials
  - Kerner, Pechmann, Spitzer (2020): Sufficient spectral gap for BEC
  - Sznitman (2023):
    - spectral gap of $-\triangle$ in a Poissonian cloud of hard spherical obstacles in large boxes in $d \geq 2$
    - for "$u = \infty$": confirms Kac–Luttinger conjecture
- We now add interparticle interaction of mean-field type (and assume $T = 0$)

Model

- $H_N^\omega = -\sum_{j=1}^{N} \triangle_j + \sum_{1\leq i<j\leq N} v_N(x_i - x_j)$ defined on $L^2((\Lambda_N^\omega)^N)$
- $\Lambda_N = (-L_N/2, L_N/2)^d$ with $L_N = \rho^{-1/d} N^{1/d}$, $\rho > 0$, $d \geq 2$
- $\Lambda_N^\omega := \Lambda_N \setminus \bigcup_m B_r(x_m^\omega)$ with Dirichlet b.c. and $\{x_m^\omega\}_m$ generated by a Poisson point process on $\mathbb{R}^d$ with constant intensity
- Λ_N^ω may consists of several components: percolation and nonpercolation regime possible!
- $v_N \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ nonnegative, even, positive-definite (i.e. $\hat{v}_N \geq 0$) such that $\hat{v}_N \in L^1(\mathbb{R}^d)$
- Thermodynamic limit $N \to \infty$ with $\frac{N}{L_N^d} = \rho > 0$

Details regarding interaction potential

- $v_N \in (L^1 \cap L^\infty)(\mathbb{R}^d)$, nonnegative, even, positive-definite such that $\hat{v}_N \in L^1(\mathbb{R}^d)$
- We assume $\|v_N\|_1 \leq \dfrac{\kappa}{N(\ln N)^{2/d}}$ for $\kappa > 0$ sufficiently small and $v_N(0) = (2\pi)^{-d/2}\|\hat{v}_N\|_1 \ll \dfrac{1}{(\ln N)^{1+2/d}}$
- Example: $v_N(x) = \dfrac{\kappa V(x)}{N(\ln N)^{2/d}}$ where $V \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ is independent of N.
- Important since in the nonpercolation regime, potential energy per particle, informally, seems to be comparable to spectral gap of the Dirichlet Laplacian $-\triangle$ on $L^2(\Lambda_N^\omega)$

Main Result

Theorem (BEC)

- *If* $\|v_N\|_1 \ll N^{-1}(\ln N)^{-2/d}$ *and* $v_N(0) \ll (\ln N)^{-(1+2/d)}$, *then*

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\frac{n_N^{1,\omega}}{N} - 1\right| < \zeta\right) = 1 \quad \forall\zeta > 0$$

i.e. (complete) BEC in probability

- $\forall\epsilon > 0\ \exists\kappa > 0$ *s.t. if* $\|v_N\|_1 \leq \kappa N^{-1}(\ln N)^{-2/d}$ *and* $v_N(0) \ll (\ln N)^{-(1+2/d)}$, *then*

$$\liminf_{N\to\infty} \mathbb{P}\left(\left|\frac{n_N^{1,\omega}}{N} - 1\right| < \zeta\right) \geq 1 - \epsilon \quad \forall\zeta > 0$$

i.e. (complete) BEC with probability almost one

Main Result: Part II

$$n_N^{1,\omega} = N \operatorname{tr}(\rho^{(1),\omega} |u_N^{\tilde{k},\omega}\rangle\langle u_N^{\tilde{k},\omega}|)$$

- $\rho^{(1),\omega}(x;y) = \int_{\mathbb{R}^{N-1}} dx_2 \dots dx_N \, \Psi_N^\omega(x, x_2, \dots, x_N)\Psi_N^\omega(y, x_2, \dots, x_N)$
- Ψ_N^ω is the N-particle ground-state of H_N^ω, $\langle \Psi_N^\omega, H_N^\omega \Psi_N^\omega \rangle = E_{\mathrm{QM},N}^{1,\omega}$
- $u_N^{\tilde{k},\omega}$ a one-particle state, the minimizer of Hartree-type functional

$$\mathcal{E}_N^{\tilde{k},\omega}[\psi] = \int_{\Lambda_N^{\tilde{k},\omega}} |\Delta\psi(x)|^2 dx + \\ + \frac{N-1}{2} \int_{\Lambda_N^{\tilde{k},\omega}} \int_{\Lambda_N^{\tilde{k},\omega}} v_N(x-y)|\psi(x)|^2|\psi(y)|^2 \, \mathrm{d}x\mathrm{d}y$$

- $\tilde{k}$ is the component on which the ground state of $-\triangle$ on $L^2(\Lambda_N^\omega)$ is supported

Outline of proof: 1. Step

$$1 - \frac{n_N^{1,\omega}}{N} \le \frac{v_N(0)}{2} \cdot \frac{1}{e_N^{2,\tilde{u},\omega} - e_N^{1,\tilde{u},\omega}}$$

where $e_N^{1,\tilde{u},\omega}$ and $e_N^{2,\tilde{u},\omega}$ are the two lowest eigenvalues of h_N^ω on $L^2(\Lambda_N^\omega)$

$$h_N^{\tilde{u},\omega} = -\triangle + (N-1)(|u_N^{\tilde{k},\omega}|^2 * v_N) - \frac{N-1}{2} \int_{\Lambda_N^\omega} \int_{\Lambda_N^\omega} v_N(x-y) |u_N^{\tilde{k},\omega}(x)|^2 |u_N^{\tilde{k},\omega}(y)|^2 \, \mathrm{d}x \mathrm{d}y$$

- $E_{\mathrm{QM},N}^{1,\omega} \le \langle u_N^{\tilde{k},\omega} \otimes \ldots \otimes u_N^{\tilde{k},\omega}, H_N^\omega \; u_N^{\tilde{k},\omega} \otimes \ldots \otimes u_N^{\tilde{k},\omega} \rangle = N e_N^{1,\tilde{u},\omega}$
- For any $\xi \in L^1(\mathbb{R}^d)$ we have (M. Lewin, 2015)

$$\sum_{1 \le i < j \le N} v_N(x_i - x_j) \ge \sum_{j=1}^{N} (\xi * v_N)(x_j) - \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v_N(x-y) \xi(x) \xi(y) \, \mathrm{d}x \mathrm{d}y - N \frac{v_N(0)}{2}$$

$\rightsquigarrow$ v_N even and positive-definite such that $\hat{v}_N \in L^1(\mathbb{R}^d)$

Outline of proof: 2. Step

$$1 - \frac{n_N^{1,\omega}}{N} \leq \frac{v_N(0)/2}{e_N^{2,\tilde{u},\omega} - e_N^{1,\tilde{u},\omega}} \leq \frac{v_N(0)/2}{e_N^{2,\omega} - e_N^{1,\omega} - (const.)N\|v_N\|_1(e_N^{1,\omega})^{d/2}}$$

where $e_N^{1,\omega}$ and $e_N^{2,\omega}$ are the two lowest eigenvalues of $-\triangle$ on Λ_N^ω

and $e_N^{1,\tilde{u},\omega}$ and $e_N^{2,\tilde{u},\omega}$ are the two lowest eigenvalues of h_N^ω on $L^2(\Lambda_N^\omega)$ where

$$h_N^{\tilde{u},\omega} = -\triangle + (N-1)(|u_N^{\tilde{k},\omega}|^2 * v_N) - \frac{N-1}{2} \int_{\Lambda_N^\omega} \int_{\Lambda_N^\omega} v_N(x-y)|u_N^{\tilde{k},\omega}(x)|^2 |u_N^{\tilde{k},\omega}(y)|^2 \,\mathrm{d}x\mathrm{d}y$$

(A little simplified:)

- $e_N^{2,\tilde{u},\omega} \geq e_N^{2,\omega}$ (i.e. interaction is ignored)
- $e_N^{1,\tilde{u},\omega} \leq \langle \varphi_N^{1,\omega}, h_N^{\tilde{u},\omega} \varphi_N^{1,\omega} \rangle$ where $\varphi_N^{1,\omega}$ is the ground-state of $-\triangle$ on Λ_N^ω

Outline of proof: 3. Step

Using $\lim_{N\to\infty} \mathbb{P}\left(e_N^{1,\omega} < (const.)(\ln N)^{-2/d}\right) = 1$,

$$1 - \frac{n_N^{1,\omega}}{N} \leq \frac{v_N(0)/2}{e_N^{2,\omega} - e_N^{1,\omega} - (const.)N\|v_N\|_1 (e_N^{1,\omega})^{d/2}}$$
$$\leq \frac{v_N(0)/2}{e_N^{2,\omega} - e_N^{1,\omega} - (const.)N\|v_N\|_1 (\ln N)^{-1}}$$

where $e_N^{1,\omega}$ and $e_N^{2,\omega}$ are the two lowest eigenvalues of $-\triangle$ on Λ_N^ω

- A.-S. Sznitman, *On the spectral gap in the Kac–Luttinger model and Bose–Einstein condensation*, Stoch. Process. Their Appl. (2023)

$$\lim_{\sigma\to 0} \liminf_{N\to\infty} \mathbb{P}\left(e_N^{2,\omega} - e_N^{1,\omega} \geq \sigma(\ln N)^{-(1+2/d)}\right) = 1$$

- We assume (i) $\|v_N\|_1 \ll N^{-1}(\ln N)^{-2/d}$ or
  (ii) $\|v_N\|_1 \leq \kappa N^{-1}(\ln N)^{-2/d}$ for $\kappa > 0$ sufficiently small,
  as well as $v_N(0) \ll (\ln N)^{-(1+2/d)}$

Remarks and Outlook I

- My be possible to relax condition of $v_N(0) \ll (\ln N)^{-(1+2/d)}$ to $v_N(0) \leq (const.)(\ln N)^{-(1+2/d)}$. However, BEC may not be complete in this case.
- We proved BEC into one-particle state $u_N^{\tilde{k},\omega}$, the minimizer of a Hartree-type functional, and where $\tilde{k}$ is the component on which the ground state of $-\triangle$ on $L^2(\Lambda_N^\omega)$ is supported
- If $\|v_N\|_1 \ll N^{-1}(\ln N)^{-2/d}$, (complete) BEC in probability also into ground-state of $-\triangle$ on $L^2(\Lambda_N^\omega)$(?)
- However, if $\|v_N\|_1 \leq \kappa N^{-1}(\ln N)^{-2/d}$ for $\kappa > 0$ sufficiently small, then (not complete) BEC with probability almost one into ground-state of $-\triangle$ on $L^2(\Lambda_N^\omega)$(?)

Remarks and Outlook II

- Comparing with [KP23] (about absence of BEC into sufficiently localized states for sufficiently strong interactions, however assumes $T > 0$ and nonpercolation regime), our condition for v_N seems quite close to being optimal.
- Therefore, for stronger interactions, BEC into a one-particle state that is not too localized?
- In noninteracting case, randomness makes BEC easier to occur and more stable, due to Lifshitz-tail behavior. However, randomness may also result in highly localized eigenfunctions. Therefore, in interacting case, randomness may hinder the occurrence of BEC and reduce its stability, at least in some sense?