Sturm-Hurwitz theorem for Quantum Graphs

Joint work with Ram Band (Technion and Potsdam)

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Theorem (Sturm 1836, rediscovered by Hurwitz, 1903)

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Theorem (Sturm 1836, rediscovered by Hurwitz, 1903)

Let $f = \sum_{k=m}^{M} a_k \sin(kx)$. Then, f has between $m - 1$ and $M - 1$ zeroes in $(0, \pi)$.

This remains true for Sturm-Liouville eigenfunctions with Dirichlet boundary conditions.

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For bounds on a_k, see Quantitative projections in the Sturm Oscillation Theorem by S. Steinerberger (2020).

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VERY FALSE: various counterexamples since the 70's.

Hybrid case: Quantum graphs

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Assumptions on Γ:

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- For other vertices, sum of inwards derivatives is zero (Neumann-Kirchhoff)
- All eigenfunctions of H_W do not vanish at any inner vertex (callde W-generic).

Let $N(f)$ be the number of inner zeroes of f.

Theorem (Band, C., 2023)

Let Γ be a W-generic graph with first Betti number β. Let f_k be the eigenfunctions of $H_W = -\frac{\partial^2}{\partial x^2} + W$ with Dirichlet boundary conditions and Neumann-Kirchhoff continuity conditions on inner vertices. Let k_i be a strictly increasing sequence and $F(x) = \sum_{i=1}^{M} a_i f_{k_i}(x)$ where each a_i is not zero. We have the following bounds:

$$k_1 - 1 - (M-1)(|V_b| + 2\beta - 2) \leq N(F)$$
$$N(F) \leq k_M - 1 + \beta + (M-1)(|V_b| + 2\beta - 2)\ .$$

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Upper bound is sharp in general

Consider $g(x, y) = \sum_{i=1}^{M} a_i e^{-\lambda_{k_i} y} f_{k_i}(x)$.

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Start at $y = -\infty$, look at nodal lines of g

Topological properties

Show that there are no isolated zeroes.

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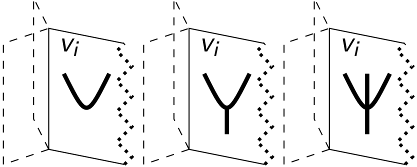
Behaviours of nodal lines:



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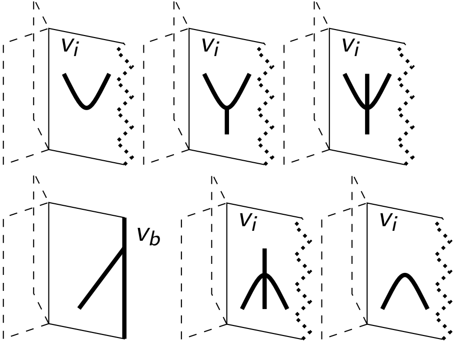

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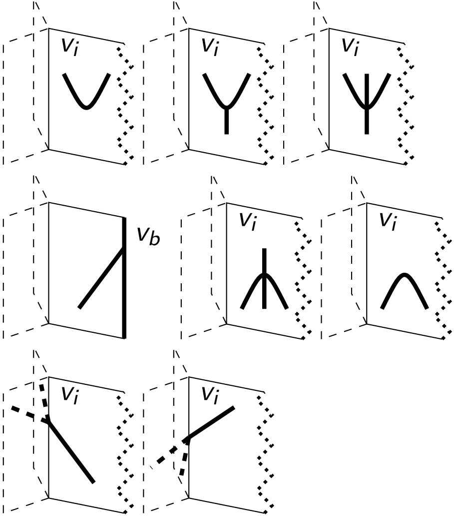

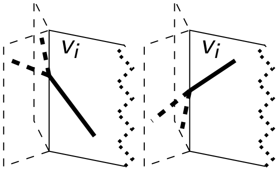
v_i
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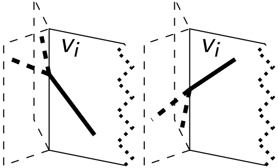
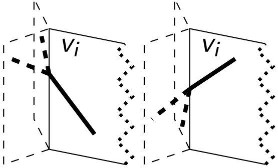


Can happen at most $M - 1$ times on each inner vertex.



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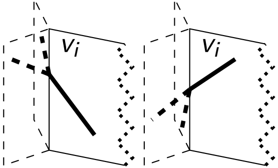
Each time it happens, it can create at most $deg(v) - 2$ new nodal lines (or reduce by that number).



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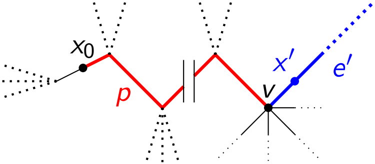
Known bounds: $k-1 \leq N(f_k) \leq k-1+\beta$ (many people)

Leads us to $k_1 - 1 - (M-1)(\sum_{inner} \deg(v) - 2) \leq N(F) \leq k_M - 1 + \beta + (M-1)(\sum_{inner} \deg(v) - 2)$.

Non-trivial examples for any tree:

x_0 is the only zero of f_2 v is the vertex of highest multiplicity p is the path between x_0 and v. x' is any point close to v that is not in v



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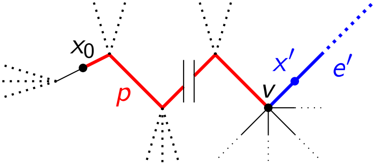

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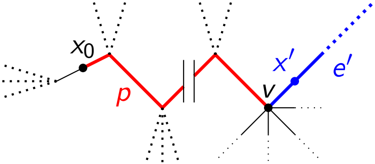

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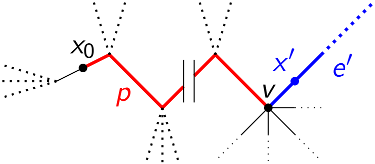

Choose a such that $af_1(x') + f_2(x') = 0$.
Choose t such that $ae^{-\lambda_1 t} f_1(v) + e^{-\lambda_2 t} f_2(v) = 0$.
Then $f(x) := ae^{-\lambda_1(t+\epsilon)} f_1(x) + e^{-\lambda_2(t+\epsilon)} f_2(x)$ has at least $deg(v) - 1$ zeroes.

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Take a very small perturbation of edge lengths to make it generic.

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- Better bounds for graphs with interesting topology.