Modelling Population Dynamics on Networks using Partial Differential Equations solved by Multigrid Methods

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Abstract

I solve partial differential equations on networks with the goal of modelling population dynamics in the Paleolithic and Neolithic, in the context of the interdisciplinary research project HESCOR[1] at the University of Cologne. Modelling human dynamics is restricted by the availability of archaeological data. In order to focus on areas where data is not too sparse for meaningful results, I restrict the domain to a network where vertices are archaeological sites, connected by edges which describe migration corridors. A method for determining these migration corridors is presented in the poster session [BP]. The actual migration dynamics are modelled using a diffusion-advection partial differential equation (PDE).

In order to formulate PDEs on graphs, I explain the network structure with the help of metric graphs. Metric graphs use an edgewise parameterization of the graph, such that differential operators can be defined on them. In order to solve the PDEs on metric graphs, I use a multigrid method [B]. The method is first developed for an elliptic PDE with Neumann-Kirchhoff conditions. The graph is discretized using a finite element (FE) discretization and a hat function basis, as described in [AB]. By combining the discretization with a weak formulation of the elliptic PDE, we can find an approximation to the solution of the PDE in the FE discretization space by solving the system of equations:

$$\begin{pmatrix} \mathbf{H}_{\mathcal{E}\mathcal{E}} & \mathbf{H}_{\mathcal{E}\mathcal{V}} \\ \mathbf{H}_{\mathcal{V}\mathcal{E}} & \mathbf{H}_{\mathcal{V}\mathcal{V}} \end{pmatrix} \mathbf{u} = \mathbf{f},$$

where $\mathbf{u}$ is the coefficient vector of the solution of the PDE written in its basis.

This system requires an efficient solver, because every edge of the metric graph is discretized using $n_e \in \mathbb{N}$ discretization points. Consequently, for sufficiently complex graphs, the discretization is applied to a large set of edges, resulting in a large system of equations. I use a multigrid method and develop suitable intergrid operators for the solution of this system of equations. Using a semi-discretization FE, this method can be extended to parabolic PDEs ([W]).

References

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[W] A. Weller, *Numerical Methods for Parabolic Partial Differential Equations on Metric Graphs*, Ph.D. Thesis, Faculty of Mathematics and Natural Sciences, University of Cologne, 2024

HESCOR Human & Earth System Coupled Research
Funded by:
Ministry of Culture and Science of the State of North Rhine-Westphalia

[1] HESCOR: Human & Earth System Coupled Research, https://hescor.uni-koeln.de