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ABSTRACT

The Ornstein-Uhlenbeck operator, a fundamental differential operator with applications in stochastic processes and partial differential equations, has been recently studied on non-compact metric star graphs by Mugnolo and Rhandi, [MR22]. Their work establishes the existence and uniqueness of classical solutions to parabolic problems with unbounded coefficients, provides an explicit representation formula for the associated semigroup, and characterizes the unique invariant measure and long-time behavior.

This thesis presents and extends the results of [MR22], focusing on the Ornstein-Uhlenbeck operator as a prime example of an unbounded operator on non-compact metric graphs. We uncover the key role of the even-odd extension method in translating results from the real line to metric star graphs, providing a powerful technique for analyzing parabolic problems in this setting.

The thesis makes several original contributions:

1. We generalize the spectral analysis to include δ-coupling vertex conditions, characterizing the eigenvalues through a transcendental equation involving the coupling strength.

2. We introduce a commutator-based approach for constructing new solutions to the associated parabolic problem, demonstrating its application to Robin boundary conditions on a semi-infinite interval.

3. We unveil the algebraic structure underlying the Ornstein-Uhlenbeck evolution operator by introducing the T-extended algebra $\mathfrak{E}_T$, which extends the Heisenberg algebra.

4. We establish an isomorphism between $\mathfrak{E}_T$ and the oscillator algebra $\mathfrak{O}$, revealing the Ornstein-Uhlenbeck evolution operator as the Casimir element of $\mathfrak{E}_T$.

These results provide a comprehensive framework for analyzing the Ornstein-Uhlenbeck operator on metric star graphs, encompassing analytical, spectral, and algebraic aspects. This work contributes to the growing body of knowledge on differential operators on metric graphs. The methods developed here may be applicable to the study of differential operators on more complex graph topologies.

ZUSAMMENFASSUNG

Der Ornstein-Uhlenbeck-Operator, ein grundlegender Differentialoperator mit Anwendungen in stochastischen Prozessen und partiellen Differentialgleichungen, wurde kürzlich von Mugnolo und Rhandi [MR22] auf nicht-kompakten metrischen Sterngraphen untersucht. Ihre Arbeit etabliert die Existenz und Eindeutigkeit klassischer Lösungen für parabolische Probleme mit unbeschränkten Koeffizienten, liefert eine explizite Darstellungsformel für die zugehörige Halbgruppe und charakterisiert das eindeutige invariante Maß sowie das Langzeitverhalten.

Diese Arbeit präsentiert und erweitert die Ergebnisse von [MR22], wobei der Fokus auf dem Ornstein-Uhlenbeck-Operator als Paradebeispiel eines unbeschränkten Operators auf nicht-kompakten metrischen Graphen liegt. Wir decken die Schlüsselrolle der Methode der geraden und ungeraden Erweiterungen bei der Übertragung von Ergebnissen von der reellen Achse auf metrische Sterngraphen auf und stellen damit eine leistungsfähige Technik zur Analyse parabolischer Probleme in diesem Kontext bereit.

Die Arbeit liefert mehrere originäre Beiträge:

1. Wir verallgemeinern die Spektralanalyse auf δ-Kopplungs-Vertexbedingungen und charakterisieren die Eigenwerte durch eine transzendente Gleichung, die die Kopplungsstärke einbezieht.

2. Wir führen einen auf Kommutatoren basierenden Ansatz zur Konstruktion neuer Lösungen des zugehörigen parabolischen Problems ein und demonstrieren dessen Anwendung auf Robin-Randbedingungen auf einem halbunendlichen Intervall.

3. Wir enthüllen die algebraische Struktur, die dem Ornstein-Uhlenbeck-Evolutionsoperator zugrunde liegt, indem wir die T-erweiterte Algebra $\mathfrak{E}_T$ einführen, die die Heisenberg-Algebra erweitert.

4. Wir etablieren einen Isomorphismus zwischen $\mathfrak{E}_T$ und der Oszillator-Algebra $\mathfrak{O}^c$ und zeigen, dass der Ornstein-Uhlenbeck-Evolutionsoperator das Casimir-Element von $\mathfrak{E}_T$ ist.

Diese Ergebnisse bilden einen umfassenden Rahmen für die Analyse des Ornstein-Uhlenbeck-Operators auf metrischen Sterngraphen und decken analytische, spektrale und algebraische Aspekte ab. Diese Arbeit erweitert das Wissen über Differentialoperatoren auf metrischen Graphen. Die hier entwickelten Methoden könnten sich auf die Untersuchung von Differentialoperatoren auf komplexeren Graphentopologien übertragen lassen.

Almost in the beginning was curiosity.

— Isaac Asimov [Asi72]

They shine for you, they shine for you
They burn for all to see
Come into these arms again
And set this spirit free

— Annie Lennox [Len]

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My journey to complete this Bachelor's thesis in Mathematics has been long and unconventional. After initially studying Physics and spending over two decades in industry, I decided to pursue my lifelong dream of obtaining a degree in Mathematics. This thesis represents the culmination of that dream's first step.

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CONTENTS

NOTATION AND CONVENTIONS

We adopt the following notational conventions throughout this thesis:

- Bold symbols (e.g., $\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{A}$) denote objects associated with the metric star graph $\boldsymbol{\mathcal{S}_m}$, including points, functions, and operators. The corresponding unbold symbols (e.g., x, f, A) refer to analogous objects on $\mathbb{R}$ or $\mathbb{R}^N$.
- Symbols in a calligraphic font (e.g., $\mathcal{A}$) represent differential expressions. Once a differential expression is restricted to a specific domain to define an operator, we denote this operator using the same letter in a non-calligraphic font (e.g., A).
- The space $L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$ denotes the L^p space on the metric star graph $\boldsymbol{\mathcal{S}_m}$ with respect to the measure $\boldsymbol{\gamma}$, while $L^p_\gamma(\mathbb{R})$ denotes the corresponding space on $\mathbb{R}$.
- The realization of an operator $\mathbf{A}$ in the space $L^p_\mu(\mathbf{S}_m)$ is denoted by $\mathbf{A}_p$, with domain $\mathrm{Dom}(\mathbf{A}_p)$.
- Specific instances of operators and spaces are distinguished by subscripts or superscripts as needed. For example, $\mathbf{A}_D$ and $\mathbf{A}_N$ denote the realizations of $\mathbf{A}$ with Dirichlet and Neumann boundary conditions, respectively.

These conventions are chosen to clarify at each point whether we are working on the star graph $\boldsymbol{\mathcal{S}_m}$ or on $\mathbb{R}$, and to distinguish between differential expressions, operators, and their realizations in various function spaces.

LIST OF ACRONYMS

OU Ornstein-Uhlenbeck

SL Sturm-Liouville

SDE Stochastic Differential Equation

BC Boundary Condition

VC Vertex Condition

LIST OF SYMBOLS

(P_Λ)	Parabolic problem associated with operator Λ, page 18
α	Multi-index for partial derivatives, page 6
Dom	Domain of an operator, page 7
γ_i	Gaussian measure on the i-th ray, page 10
$\hat{\boldsymbol{u}}^n$	Function defined on $[0,\infty) \times \boldsymbol{\mathcal{S}}_{\boldsymbol{m}}^{\boldsymbol{n}}$ from solutions of truncated problems, page 22
$\kappa(t,x,y)$	Integral kernel of the semigroup generated by $\widetilde{L}$, page 19
$\kappa_{OU}(t,x,y)$	Ornstein-Uhlenbeck kernel on $\mathbb{R}$, page 25
$\langle \cdot, \cdot \rangle$	Standard scalar product in $\mathbb{R}^N$, page 4
$\langle \cdot, \cdot \rangle_{L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}}_{\boldsymbol{m}})}$	Scalar product on $L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}}_{\boldsymbol{m}})$, page 13
$\mathbb{1}$	Constant function equal to 1 everywhere, page 24
$\mathbb{C}$	Complex number set, page 5
$\mathbb{C}^-$	Left half-plane of the complex plane, page 5
$\mathbb{K}$	Field of real or complex numbers, page 12
$\mathbb{N}$	Set of natural numbers, page 9
$\mathbb{N}_0$	Set of natural numbers including zero, page 6
$\mathbf{0}$	Central vertex of the metric star graph, page 3
$\mathcal{C}(\mathbf{0})$	Set of conditions on the central vertex, page 3
$\mathcal{D}_\nu^q$	Function space of solutions to the Ornstein-Uhlenbeck equation, page 46
$\mathcal{I}$	Identity operator or matrix, depending on context, page 4
$\mathfrak{E}_T$	T-extended algebra, page 53
$\mathfrak{O}^c$	Oscillator algebra, page 55
ω_{ijk}	Constants defining the operator H_Ω, page 46
$\mathcal{A}$	Ornstein-Uhlenbeck differential operator, page 1
∂	Annihilation operator, page 9
∂^*	Creation operator, page 9
ϕ_i	Canonical injection from $[0,\infty)$ to $[0,\infty) \times i$, page 10
$\boldsymbol{\gamma}$	Pushforward measure on $\boldsymbol{\mathcal{S}}_{\boldsymbol{m}}$, page 11

$\boldsymbol{\mathcal{L}}$	Kolmogorov differential operator on metric star graphs, page 17
$\boldsymbol{\Sigma}$	Pushforward sigma-algebra on $\boldsymbol{\mathcal{S}_m}$, page 11
$\sim$	Equivalence relation for gluing rays at the origin, page 10
$\boldsymbol{A}$	Ornstein-Uhlenbeck operator on the metric star graph $\boldsymbol{\mathcal{S}_m}$, page 26
$\boldsymbol{A}_2$	Realization of the Ornstein-Uhlenbeck operator in $L^2_{\boldsymbol{\gamma}_m}(\boldsymbol{\mathcal{S}_m})$, page 29
$\boldsymbol{L}$	Operator associated with $\boldsymbol{\mathcal{L}}$ with specified domain, page 18
$\boldsymbol{OU_m}(t)$	Ornstein-Uhlenbeck semigroup on the metric star graph $\boldsymbol{\mathcal{S}_m}$, page 26
$\boldsymbol{r_i}$	Ray $[0,\infty) \times i$ of the star graph, page 10
$\boldsymbol{T_m}(t)$	Semigroup generated by $\boldsymbol{L}$ on $C_b(\boldsymbol{\mathcal{S}_m})$, page 19
Re	Real part of a complex number, page 5
$\boldsymbol{\mathcal{S}_m}$	Metric star graph with m half-lines, page 3
$\boldsymbol{\mathcal{S}_m^n}$	Truncated star graph, page 16
Σ	Sigma-algebra on S_m, page 11
$\sigma(A_p)$	Spectrum of A_p, page 8
$\sigma(B)$	Set of eigenvalues of matrix B, page 5
$\Box_{\mathcal{A}}$	Ornstein-Uhlenbeck evolution operator, page 46
τ	Exponential decay factor, defined as e^{-t}, page 25
$\widetilde{f_i}$	Extended function on $\mathbb{R}$ associated with f_i, page 22
$\widetilde{L}$	Extended operator on $\mathbb{R}$ associated with $\boldsymbol{L}$, page 18
$\widetilde{W^{k,p}_{\text{loc}}}(\boldsymbol{\mathcal{S}_m})$	Direct sum of local Sobolev spaces on rays of $\boldsymbol{\mathcal{S}_m}$, page 18
$\boldsymbol{x}_i$	Point (x,i) in $\boldsymbol{\mathcal{S}_m}$, page 10
*	Superscript denoting the adjoint of an operator or matrix, page 4
A_p	Generator of the OU semigroup on $L^p_\gamma(\mathbb{R}^N)$, page 7
B	Non-trivial matrix with real-valued coefficients in the OU operator definition, page 3
$C(\boldsymbol{\mathcal{S}_m})$	Space of continuous functions on $\boldsymbol{\mathcal{S}_m}$, page 12
$C^\nu_{\text{loc}}([0,\infty))$	Space of locally Hölder continuous functions with exponent ν, page 17
C_0-semigroups	Strongly continuous semigroups, page 27
$C_b(\mathbb{R}^N)$	Space of continuous and bounded functions on $\mathbb{R}^N$, page 5
$C_b(\boldsymbol{\mathcal{S}_m})$	Space of bounded and continuous functions on $\boldsymbol{\mathcal{S}_m}$, page 12
$C_c^\infty(\mathbb{R}^N)$	Space of smooth functions with compact support on $\mathbb{R}^N$, page 9

D_α	Partial derivative operator corresponding to multi-index α, page 6
d_i	Euclidean distance on the i-th ray, page 10
$d_{\boldsymbol{\mathcal{S}_m}}$	Path metric on $\boldsymbol{\mathcal{S}_m}$, page 11
d_{S_m}	Metric of disjoint union on S_m, page 10
f_i	Restriction of $\boldsymbol{f}$ to the i-th ray, page 12
G	Generic Quantum Graph, page 3
G_{OU}	Ornstein-Uhlenbeck quantum graph triple, page 3
h	Coupling parameter, page 45
H_Ω	General first-order differential operator commuting with $\Box_{\mathcal{A}}$, page 46
H_Ω^n	Higher-order operator commuting with $\Box_{\mathcal{A}}$, page 53
I	Set of indices $\{1, \dots, m\}$ for the rays of the star graph, page 10
L^2_{even}	Subspace of even functions in $L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m})$, page 30
L^2_{odd}	Subspace of odd functions in $L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m})$, page 30
$L^p(\mathbb{R}^N)$	Lebesgue space of p-integrable functions on $\mathbb{R}^N$, page 6
$L^p_\gamma(\mathbb{R}^N)$	Gaussian L^p space on $\mathbb{R}^N$, page 6
$L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$	Gaussian L^p space on $\boldsymbol{\mathcal{S}_m}$, page 12
m	Number of half-lines in the metric star graph $\boldsymbol{\mathcal{S}_m}$, page 3
$OU(t)$	Ornstein-Uhlenbeck semigroup, page 4
Q	Real symmetric and positive definite matrix in the OU operator definition, page 3
q	Canonical surjection from S_m to $\boldsymbol{\mathcal{S}_m}$, page 11
Q_∞	Limit of $t_{B,Q}$ as t approaches infinity, page 5
$s(B)$	Spectral bound of matrix B, page 8, 9
$S(t)$	Heat semigroup, page 4
S_m	Disjoint union of m copies of $[0, \infty)$, page 10
T	Time derivative operator, page 53
$T(t)$	Semigroup of linear operators on $C_b(\mathbb{R})$ associated with $\widetilde{L}$, page 24
T_t	General notation for a semigroup at time t, page 7
$t_{B,Q}$	Time-dependent matrix in the OU semigroup representation, page 4
$W^{k,p}_\gamma(\mathbb{R}^N)$	Gaussian Sobolev space on $\mathbb{R}^N$, page 6
$W^{k,p}_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$	Gaussian Sobolev space on $\boldsymbol{\mathcal{S}_m}$, page 13

W_t Standard n-dimensional Brownian motion, page 5

X_t Time-dependent operator in the extended algebra $\mathfrak{E}_T$; analogous to annihilation operator, page 53

X_t^* Time-dependent operator in the extended algebra $\mathfrak{E}_T$; analogous to creation operator, page 53

1 INTRODUCTION

Desocupado lector: sin juramento me podrás creer
que quisiera que este libro, como hijo del entendimiento,
fuera el más hermoso, el más gallardo
y más discreto que pudiera imaginarse. [1]

— Miguel de Cervantes Saavedra [CS05]

1.1 BACKGROUND AND MOTIVATION

The present work investigates the behavior of the Ornstein-Uhlenbeck (OU) operator when acting on functions defined on *metric star graphs* consisting of a finite number of half-lines. This scenario serves as one of the simplest non-trivial examples of an unbounded operator acting on non-compact graphs.

The OU differential operator, denoted by $\mathcal{A}$, is a fundamental differential operator connected to the theory of stochastic processes, functional analysis, and partial differential equations. Its study has developed a rich theory concerning domains, spectra, and functional inequalities.

One-dimensional OU operators are defined on suitable functions f by

$$\mathcal{A}f := \frac{1}{2}\frac{d^2 f}{dx^2} - x\frac{df}{dx}, \quad x \in \mathbb{R}, \tag{1.1}$$

and are named after L. Ornstein and G. Uhlenbeck [UO30]. Their work on stochastic motion led to the OU process, which offers a framework where trajectories possess finite velocities and stationary distributions, notably the standard Gaussian measure.

In recent years, the study of the OU operator has led to significant advances in the theory of functional inequalities [BGL14], underpinning the analysis of diverse mathematical models in statistical mechanics, quantum field theory, and information theory [Mar+21].

1.2 OPERATORS ON METRIC GRAPHS

The study of operators on metric graphs, or quantum graphs, inherently connects the realms of discreteness and continuity, finding applications in both physics and pure mathematics. In physics, these operators have elucidated phenomena in quantum systems [Man10], Anderson localization [And58], and the properties of nanostructures [Bad+07]. Mathematically, they provide a rich setting for exploring the interplay between graph topology and spectral properties of differential operators [Ber16].

1 Idle reader: thou mayest believe me without any oath
that I would this book, as it is the child of my brain,
were the fairest, gayest, and cleverest that could be imagined. [CS04]

In the context of our study, the OU operator on metric star graphs represents a natural progression in this field. While much of the existing literature focuses on the Laplacian and Schrödinger operators, investigating the OU operator on metric graphs opens new research directions. It combines the rich structure of quantum graphs with the stochastic interpretation and functional analytic properties of the OU operator.

The work of Mugnolo and Rhandi [MR22], which forms a cornerstone of this thesis, represents one of the first systematic studies of the OU operator on metric star graphs.

1.3 STRUCTURE OF THE THESIS AND ORIGINAL CONTRIBUTIONS

This thesis is structured as follows:

- Chapter 2 establishes fundamental definitions and properties related to the OU operator, function spaces, and metric graphs.
- Chapter 3 presents key results by Mugnolo and Rhandi [MR22] on the OU operator applied to non-compact metric star graphs.
- Chapter 4 explores the connection between the OU operator and a singular Sturm-Liouville problem, extending spectral results to δ–Coupling Vertex Conditions.
- Chapter 5 introduces a commutator-based approach for constructing solutions to the associated parabolic problem under different boundary conditions.
- Chapter 6 unveils the algebraic structure underlying the OU evolution operator, introducing the *T-extended algebra* $\mathfrak{E}_T$ and establishing its isomorphism with the *oscillator algebra*.

This work aims to extend our understanding of the OU operator on metric star graphs, offering insights into its analytical, spectral, and algebraic properties. By building upon existing research, we hope to contribute to the ongoing study of differential operators on metric graphs. The methods and results presented here may prove useful in future investigations of related problems, potentially including more complex graph topologies and other operator classes.

2 PRELIMINARIES

"Begin at the beginning," the King said gravely,
"and go on till you come to the end: then stop."

— Lewis Carroll [Car65]

This thesis deals with differential operators on metric graphs, often referred to as *quantum graphs* in the literature. A quantum graph G is characterized by a triple $(\mathcal{G}, \mathcal{D}, \mathcal{C}(v))$ [Kur23] consisting of:

- a metric graph $\mathcal{G}$,
- a differential operator $\mathcal{D}$,
- a set of vertex conditions $\mathcal{C}(v)$, linear relations that couple the values and derivatives of the functions at the graph's vertices. The choice of the local vertex conditions decides whether the operator is self-adjoint and the behavior of its spectrum.

We will use the term "metric graph" to refer to $\mathcal{G}$ and reserve the term "quantum graph" for the entire triple G when necessary—though the latter will be rarely used.

Specifically, our focus is on the quantum graph $G_{OU} := (\mathcal{S}_{\boldsymbol{m}}, \mathcal{A}, \mathcal{C}(\mathbf{0}))$, where:

- $\mathcal{S}_{\boldsymbol{m}}$ is a metric star graph with $m \in \mathbb{N}$ half-lines,
- $\mathcal{A}$ is the Ornstein-Uhlenbeck differential operator,
- $\mathcal{C}(\mathbf{0})$ is a set of conditions on the central vertex $\mathbf{0}$ of the metric star graph.

We proceed to provide detailed definitions for each element of the triple.

2.1 THE ORNSTEIN-UHLENBECK DIFFERENTIAL OPERATOR $\mathcal{A}$

The OU operator stands as a prominent example among elliptic operators with unbounded coefficients.

It is defined *ab initio* on smooth functions[1] φ by

$$\begin{aligned}(\mathcal{A}\varphi)(x) &= \frac{1}{2}\sum_{i,j=1}^{N} q_{ij} D_{ij}\varphi(x) + \sum_{i,j=1}^{N} b_{ij} x_j D_i \varphi(x) &\quad (2.1)\\ &= \frac{1}{2}\operatorname{Tr}(QD^2\phi)(x) + \langle Bx, D\varphi(x)\rangle, \quad x \in \mathbb{R}^N, &\quad (2.2)\end{aligned}$$

where $Q = (q_{ij})_{i,j=1,\dots,N}$ is a real symmetric and positive definite matrix and $B = (b_{ij})_{i,j=1,\dots,N}$ is non-trivial matrix with real-valued coefficents. We assume Q is

1 We will show in Section 2.1.3, that the domain of the OU operator is much larger.

strictly positive, as it is later inverted in the definition of the associated semigroup and the invariant measure. The associated semigroup $OU(t)$ admits an explicit representation formula due to Kolmogorov [Kol31]

$$\big(OU(t)f\big)(x) := \frac{1}{(2\pi)^{N/2}(\det t_{B,Q})^{1/2}} \int_{\mathbb{R}^N} e^{-\langle t_{B,Q}^{-1} y, y\rangle/2} f(x_{tB} - y)\,\mathrm{d}y, \tag{2.3}$$

where x_{tB} is defined as

$$x_{tB} := e^{tB}x,$$

$t_{B,Q}$ is defined as:

$$t_{B,Q} := \int_0^t e^{sB} Q e^{sB^*}\,\mathrm{d}s,$$

and B^* denotes the adjoint matrix of B.

The OU semigroup $OU(t)$, defined in (2.3), admits a simple representation in terms of *Gaussian distributions* $\mathcal{N}(l, Q)$. Recall that $\mathcal{N}(l, Q)$ denotes the probability measure on $\mathbb{R}^N$ with density $\gamma_{l,Q}$ given by

$$\gamma_{l,Q}(x) := \frac{1}{(2\pi)^{N/2}(\det Q)^{1/2}} e^{\frac{-\langle Q^{-1}(x-l),(x-l)\rangle}{2}}, \tag{2.4}$$

where Q is an $N \times N$ positive definite matrix, $l \in \mathbb{R}^N$, and $\langle \cdot, \cdot \rangle$ represents the standard scalar product in $\mathbb{R}^N$.

The action of $OU(t)$ on a function f in the appropriate space (see Section 2.1.3) can then be written concisely as

$$\big(OU(t)f\big)(x) = \int_{\mathbb{R}^N} f(y)\gamma_{x_{tB},t_{B,Q}}(y)\,\mathrm{d}y. \tag{2.5}$$

A compelling perspective on the OU semigroup emerges when considering its relation to the *heat semigroup* $S(t)$ defined by

$$\big(S(t)f\big)(x) := \int_{\mathbb{R}^N} f(y)\gamma_{x,t\mathcal{I}}(y)\,\mathrm{d}y, \tag{2.6}$$

where $\mathcal{I}$ is the identity matrix and $\gamma_{x,t\mathcal{I}}$ the *heat kernel*.

Comparing the integral kernels in equations (2.5) and (2.6), we observe that $OU(t)$ acts like a heat semigroup under two transformations:

1. A matrix-valued time transformation: $t\mathcal{I} \mapsto t_{B,Q}$.
2. An exponential drift transformation: $x \mapsto x_{tB}$.

Remark 2.1 *In the one-dimensional case, this relationship takes the form:*

$$\big(OU(t)f\big)(x) = \big(S(t_{B,Q})f\big)(x_{tB}). \tag{2.7}$$

For reference, we provide the specific values of the parameters for $N = 1$, which constitutes the primary focus of this thesis:

$$Q = (1),\ B = (-1),\ x_{tB} = e^{-t}x,\ t_{B,Q} = \int_0^t e^{-s^2}\,\mathrm{d}s. \tag{2.8}$$

The following sections present well-known properties of this operator and its associated semigroup in the $\mathbb{R}^N$ setting.

2.1.1 *Invariant Measure*

Definition 2.2 (Invariant measure) *An* invariant measure γ *for* $\big(OU(t)\big)_{t\geq 0}$ *is a probability measure on* $\mathbb{R}^N$ *such that*

$$\int_{\mathbb{R}^N} (OU(t)\phi)(x)\gamma(\mathrm{d}x) = \int_{\mathbb{R}^N} \phi(x)\gamma(\mathrm{d}x),$$

holds for every $t \geq 0$ *and* $\phi \in C_b(\mathbb{R}^N)$*, the space of all continuous and bounded functions in* $\mathbb{R}^N$*.*

In [DPZ92, Theorem 11.7] it is proved that the condition

$$\sigma(B) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\} =: \mathbb{C}^-$$

if and only if there exists an invariant measure γ for $\big(OU(t)\big)_{t\geq 0}$. Here, $\sigma(B)$ denotes the set of eigenvalues of B. In our setting (2.8), we have $B = (-1)$, which satisfies this condition, thus ensuring the existence of an invariant measure.

Moreover, this invariant measure γ is unique and is given by the non-degenerate Gaussian distribution $\mathcal{N}(0, Q_\infty)$, [Lun97, Lemma 3.1], with density γ_{0,Q_∞} given by:

$$\gamma_{0,Q_\infty}(x) = \frac{1}{(2\pi)^{N/2}(\det Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1}x,x\rangle/2}$$

where

$$Q_\infty := \lim_{t\to\infty} t_{B,Q} = \int_0^\infty e^{sB} Q e^{sB^*}\,\mathrm{d}s.$$

2.1.2 *Connection to Probability*

Operators with unbounded coefficients frequently emerge in the context of stochastic perturbations of ordinary differential equations. Consider the ordinary differential equation $u'(t) = Bu(t)$ in $\mathbb{R}^n$. When perturbed by noise $\sqrt{Q}\mathrm{d}W_t$ (W_t being a standard n-dimensional Brownian motion), yields the Stochastic Differential Equation (SDE):

$$\mathrm{d}X = BX\mathrm{d}t + Q^{1/2}\mathrm{d}W_t, \tag{2.9}$$

The semigroup $OU(t)$ defined in (2.3) is the associated Markov semigroup, satisfying:

$$\big(OU(t)\phi\big) = \mathbb{E}[\phi(X(t,x))], \tag{2.10}$$

for a broad class of initial conditions ϕ. Here, $\mathbb{E}$ denotes the expectation for the Brownian motion's probability measure [Lun97].

The stochastic perspective offers several advantages in studying the OU operator and its associated semigroup. Firstly, it provides a probabilistic interpretation of the semigroup action, enabling using tools from stochastic analysis to investigate its properties [DPZ92]. Secondly, the stochastic formulation allows for the derivation of explicit solutions to the OU equation in terms of stochastic integrals, which can be valuable in understanding the behavior of the semigroup [KS91]. Moreover, the connection to SDEs facilitates the study of the long-time behavior of the OU process, including the existence and uniqueness of invariant measures and the convergence of the semigroup to some equilibrium [Bog18; MPP02].

Note that (2.9) coincides in one dimension with the OU process (see Figure 2.1),

$$\mathrm{d}x_t = -\theta x_t \,\mathrm{d}t + \sigma \mathrm{d}W_t,$$

where $\theta, \sigma > 0$ and W_t denotes the Wiener process.

Historically, its primary application in physics was to model the velocity of a massive Brownian particle subject to friction. While the original paper [UO30] does not explicitly present the SDE form, it establishes the Langevin equation, which can be transformed into the above SDE.

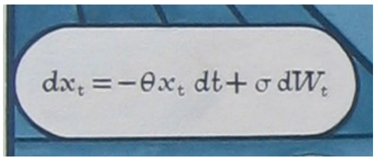

Figure 2.1: Simplified formula for the Ornstein–Uhlenbeck process, depicted in a mural in Oosterkade, The Netherlands, near Ornstein's laboratory. Source: [Orn].

2.1.3 *Domain of the realizations of $\mathcal{A}$ and $OU(t)$ in $\mathbb{R}^N$.*

We employ the Gaussian measure to construct function spaces that will serve as the domains of the realizations of the operator $\mathcal{A}$ and the semigroup $OU(t)$ (see [Lun97, Theorem 4.1]). The choice of the Gaussian measure is motivated by two key properties. First, the OU operator admits a spectral decomposition in spaces with the Gaussian measure, with the Hermite polynomials as eigenfunctions and the non-negative integers as the corresponding eigenvalues. This explicit spectral representation simplifies the analysis of the operator and its properties. Moreover, as discussed in the previous subsection, the Gaussian measure is invariant under the OU semigroup, simplifying the study of the operator's long-time behavior.

Definition 2.3 *For $1 \leq p < \infty$ and $\gamma := \gamma_{m,Q}(x)$ the Gaussian Lebesgue space $L^p_\gamma(\mathbb{R}^N)$ is defined by*

$$L^p_\gamma(\mathbb{R}^N) := \left\{ f : \mathbb{R}^N \to \mathbb{K} \text{ measurable} \,\middle|\, f\gamma^{1/p} \in L^p(\mathbb{R}^N) \right\}.$$

Definition 2.4 *For $s > 0$, the Gaussian Sobolev space $W^{k,p}_\gamma(\mathbb{R}^N)$, is defined as*

$$W^{k,p}_\gamma(\mathbb{R}^N) := \left\{ f \in L^p_\gamma(\mathbb{R}^N) \,\middle|\, D_\alpha f \in L^p_\gamma(\mathbb{R}^N),\ |\alpha| \leq k \right\},$$

where $k \in \mathbb{N}_0, 1 \leq p < \infty$, $W^{0,p}_\gamma(\mathbb{R}^N) := L^p_\gamma(\mathbb{R}^N)$, α is a multi index, and $D_\alpha f$ is understood as a weak derivative of f.

The following theorem, due to Metafune *et al.* [Met+02], characterizes the generator A_p of the OU semigroup $OU(t)$ on the Gaussian L^p space on $\mathbb{R}^N$. Specifically, it establishes that A_p is the realization of the OU differential operator $\mathcal{A}$ on the Gaussian Sobolev space $W^{2,p}_\gamma(\mathbb{R}^N)$.

Theorem 2.5 [Met+02, Theorem 3.4] *Let Q be a real, symmetric, positive definite matrix, and B a real matrix with all its eigenvalues in the left half-plane. Then, for $1 < p < \infty$ the generator A_p of the Ornstein-Ulenbeck semigroup $OU(t)$ (2.3) on $L^p_\gamma(\mathbb{R}^N)$ is the OU Operator $\mathcal{A}$ (2.1) acting on $W^{2,p}_\gamma(\mathbb{R}^N)$. In particular*

$$\mathrm{Dom}(A_p) = W^{2,p}_\gamma(\mathbb{R}^N). \tag{2.11}$$

The proof of this theorem is omitted here as it requires a detailed analysis of the semigroup properties and the use of interpolation theory, which is beyond the scope of this work.

2.1.4 *Properties of $OU(t)$*

The semigroup $OU(t)$ (see 2.3) is a strongly continuous contraction semigroup in $L^p_\gamma(\mathbb{R}^N)$, $p \in [1, \infty)$, i. e.,

$$T_{t+s} = T_t \circ T_s, \quad \|T_t f\|_p \leq \|f\|_p \quad \lim_{t\to 0} \|T_t f - f\|_p = 0,$$

for all $f \in L^p_\gamma(\mathbb{R}^N)$, and all $s, t > 0$. Furthermore, it preserves positivity, meaning it preserves the cone of non-negative functions in the L^p spaces. More precisely, for any non-negative initial data $f_0 \in L^p_\gamma(\mathbb{R}^N)$, the solution $u(t, x) := \big(OU(t)f_0\big)(x)$ remains non-negative for all $t > 0$ and $x \in \mathbb{R}^N$, [Bog18, Theorem 1.1].

It is differentiable and compact for $p \in (1, \infty)$ and assuming Q is strictly positive, $OU(t)$ is analytic in $L^p_\gamma(\mathbb{R}^N)$ for $p \in (1, \infty)$, [MPP02], [LB06].

Here, we say that the semigroup $OU(t)$ is:

- **Differentiable** if for every $f \in L^p_\gamma(\mathbb{R}^N)$, the map $t \mapsto OU(t)f$ is differentiable as a function from $(0, \infty)$ to $L^p_\gamma(\mathbb{R}^N)$.
- **Compact** if for every $t > 0$, the operator $OU(t)$ maps bounded sets in $L^p_\gamma(\mathbb{R}^N)$ into relatively compact sets in $L^p_\gamma(\mathbb{R}^N)$.
- **Analytic** if for every $f \in L^p_\gamma(\mathbb{R}^N)$, the map $t \mapsto OU(t)f$ admits an analytic extension to a sector $z \in \mathbb{C} \setminus 0 : |\arg z| < \delta$ for some $\delta \in (0, \frac{\pi}{2}]$.

The following table 2.1 summarises these properties with respect to the underlying space.

2.1.5 *Spectrum of A_p for $p \in [1, \infty)$.*

The compactness of $OU(t)$ in $L^p_\gamma(\mathbb{R}^N)$ for any $p \in (1, \infty)$, yields a discrete spectrum for its generator A_p. We rely on [MPP02] to obtain a complete spectrum characterisation.

	$L^1_\gamma(\mathbb{R}^N)$	$L^p_\gamma(\mathbb{R}^N)$	$L^\infty_\gamma(\mathbb{R}^N)$
strong continuity	✓	✓	✗
contraction	✓	✓	✓
compactness	✗	✓	✗
differentiability	✗	✓	✗
analyticity	✗	✓✗(*)	✗

Table 2.1: Properties of the semigroup $OU(t)$, $1 < p < \infty$.
(*) ✓ for Q invertible, ✗ otherwise.

Theorem 2.6 [MPP02, Theorems 3.1 and 5.1] *Let $\lambda_1, \dots, \lambda_r$ be the (distinct) eigenvalues of B, then for $p \in (1, \infty)$,*

$$\sigma(A_p) = \left\{ \lambda = \sum_{j=1}^{r} n_j \lambda_j \ : \ n_j \in \mathbb{N} \right\}.$$

Moreover, the linear span of the generalized eigenfunctions of A_p is dense in $L^p_\gamma(\mathbb{R}^N)$.

Every eigenfunction of A_p is a polynomial whose degree is bounded above by $(\operatorname{Re}\lambda)/s(B)$, with $s(B)$ being the spectral bound *of the matrix B, defined as*

$$s(B) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(B)\}, \tag{2.12}$$

where $\sigma(B)$ denotes the spectrum of B.

In the particular case that $Q = I, B = -I$, the spectrum in $L^p_\gamma(\mathbb{R}^N)$, $p \in (1, \infty)$, is the set of negative integers with the (multidimensional) Hermite polynomials forming a complete orthonormal basis of eigenfunctions. In other words, the span of the Hermite polynomials is dense in the function space $L^p_\gamma(\mathbb{R}^N)$.

For $p = 1$, we have that the spectrum of A_1 is the left half-plane. Thus, each complex number λ with $\operatorname{Re}\lambda < 0$ is an eigenvalue.

We omit the proof of this theorem as it follows from standard methods in spectral theory.

Of particular note is the spectral characterization presented above, which reveals the fundamental role of Hermite polynomials in the theory of OU operators, especially in the case where $Q = I$ and $B = -I$. Beyond its intrinsic interest, this case serves as a prototype for understanding the general structure of OU spectra.

2.1.6 *Hermite polynomials*

We now define Hermite polynomials and establish their fundamental properties in relation to the one-dimensional OU operator. This connection, following [LB06], characterizes the spectral structure of $\mathcal{A}$ through creation and annihilation operators.

We begin by defining the one-dimensional Hermite polynomial for $x \in \mathbb{R}$:

$$H_n(x) := (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}, \quad x \in \mathbb{R}.$$

Next, we introduce the annihilation ∂ and creation ∂^* operators in $W^{1,2}_\gamma(\mathbb{R}^N)$:

$$(\partial\phi)(x) := \phi'(x), \quad (\partial^*\phi)(x) := -\phi'(x) + 2x\phi(x), \quad \phi \in W^{1,2}_\gamma(\mathbb{R}^N). \tag{2.13}$$

Note that ∂^* is the *formal adjoint* to ∂ since

$$\int_{\mathbb{R}^N} \partial\phi\, \psi\, \gamma(\mathrm{d}x) = \int_{\mathbb{R}^N} \phi\, \partial^*\psi\, \gamma(\mathrm{d}x), \quad \text{for all } \phi, \psi \in W^{1,2}_\gamma(\mathbb{R}^N),$$

which follows by integrations by parts, if $\phi, \psi \in C^\infty_c(\mathbb{R}^N)$, and can be extended to the entire space $W^{1,2}_\gamma(\mathbb{R}^N)$ using a density argument based on the fact that $C^\infty_c(\mathbb{R}^N)$ is dense in $W^{1,2}_\gamma(\mathbb{R}^N)$, (see [LB06]).

With these operators, the one-dimensional OU operator $\mathcal{A}$ ((1.1)) can be expressed as:

$$\mathcal{A} = -\frac{1}{2}\partial^*\partial.$$

This formulation reveals $\mathcal{A}$ as the natural analogue of the Laplacian in Gaussian space $L^2_\gamma(\mathbb{R}^N)$, mirroring the structure of $\Delta = -D \cdot D^*$ in Euclidean space, where D is a differential operator and D^* its adjoint.

The representation of $\mathcal{A}$ using the creation and annihilation operators also provides a quantum mechanical perspective on the operator and its eigenfunctions. It also simplifies the analysis of the operator's spectral properties and the action of the associated semigroup on the Hermite polynomials.

We now summarize the key properties of Hermite polynomials, which are generally straightforward to verify:

Lemma 2.7 [LB06, Proposition 9.3.27] *Consider the Hermite polynomias $H_n(x)$, where $n \in \mathbb{N} \cup \{0\}$. These polynomials satisfy the following properties:*

(i) (Recurrence relation) *For $n \geq 1$, $H_n = \partial^* H_{n-1}$, with the base case $H_0 = 1$.*

(ii) (Degree and leading coefficient) *H_n is a polynomial of degree n with leading coefficient 2^n.*

(iii) (Derivative as ladder operator) $\partial H_{n+1} = 2(n+1)H_n$.

(iv) (Eigenvalues) $\mathcal{A}H_n = -nH_n$.

(v) (Orthonormality) $\left\{\frac{1}{\sqrt{2^n n!}}H_n\right\}_{n\in\mathbb{N}}$ *is an orthonormal basis of $L^2_\gamma(\mathbb{R})$.*

The spectral properties of the OU operators are fundamental for analyzing various problems on metric spaces, particularly on metric star graphs. The next section thoroughly defines metric star graphs and introduces the necessary function spaces and measures.

2.2 METRIC STAR GRAPHS

Having explored the OU differential operator $\mathcal{A}$, we now turn our attention to the first element of the quantum graph triple $G_{OU} = (\boldsymbol{\mathcal{S}_m}, \mathcal{A}, \mathcal{C}(\mathbf{0}))$: the metric star graph $\boldsymbol{\mathcal{S}_m}$. Recall that a metric star graph consists of a single central vertex $\mathbf{0}$ and a set of m rays, $m \in \mathbb{N}$ emanating from it. In this work, we will consider each ray to be an infinite half-line that starts at the central vertex end $\mathbf{0}$ and extends indefinitely in one direction.

To establish the definition of a metric star graph, we adhere to the formalism outlined in [Mug21], permitting the extension of both a metric and a measure from (semi-infinite) intervals to a star graph.

Let $I := \{1, \dots, m\}$[2], for $m \in \mathbb{N}$. First, we consider the collection $[0, \infty)_{i \in I}$, of positive half-lines endowed with the Euclidean distance $d_i := |\cdot|$ and Gaussian measure γ_i with density $\gamma_i(x) := \frac{2}{m\sqrt{\pi}} e^{-x^2}$, and we construct the disjoint union S_m of the intervals:

$$S_m := \bigsqcup_{i \in I} [0, \infty) = \bigcup_{i \in I} ([0, \infty) \times \{i\})$$

Definition 2.8 *The half-line* $[0, \infty) \times \{i\}$ *is called a* ray *and will be denoted with* $\boldsymbol{r_i}$.

Next, we need to endow S_m with a metric. We follow [BS04, Def. 3.1.15] and define the metric of disjoint union d_{S_m} on S_m as

$$d_{S_m}\big((x,i),(y,j)\big) := \begin{cases} d_i(x,y) = |x-y|, & \text{if } i = j, \\ \infty, & \text{otherwise.} \end{cases}$$

where we also follow [BS04] by adopting the generalized notion of distance that allows for the value ∞.

The topology induced by the d_{S_m} metric agrees with the disjoint union topology of S_m [Mug21] where a subset U of S_m is open in S_m if and only if its preimage under the canonical injection $\phi_i \colon [0, \infty) \hookrightarrow [0, \infty) \times \{i\}$ is open in $[0, \infty)$ for each $i \in I$.

We now glue together all the *left ends* $\{(0, i) : i \in I\}$ of the rays $\boldsymbol{r_i}$. This is achieved by using the equivalence relation $\sim$ on S_m defined as

$$(x,i) \sim (y,j) \iff (x = y \text{ and } i = j) \text{ or } (x = y = 0)$$

Definition 2.9 (Star graph) *A star graph of m rays, denoted by* $\boldsymbol{\mathcal{S}_m}$*, is a space constructed from the disjoint union of m copies of the interval* $[0, \infty)$ *glued together according to the* $\sim$ *equivalence relation. Formally,*

$$\boldsymbol{\mathcal{S}_m} := S_m / \sim = \left(\bigsqcup_{i \in I} [0, \infty) \right) \Big/ \sim \tag{2.14}$$

Figure 2.2 illustrates this construction for $m = 5$, showing both the disjoint union S_5 and the resulting star graph $\boldsymbol{\mathcal{S}_5}$.[3]

We denote points in $\boldsymbol{\mathcal{S}_m}$ as $\boldsymbol{x}_i := (x, i)$[4], where $x \in [0, \infty)$ and $i \in I$. The absolute value $|\cdot|$ on $\boldsymbol{\mathcal{S}_m}$ is defined as the projection onto the first component, i. e., $|\boldsymbol{x}_i| := x$. It follows directly that $|\boldsymbol{x}_i| \geq 0$ for all $\boldsymbol{x}_i \in \boldsymbol{\mathcal{S}_m}$.

We use the notation $\mathbf{0} := [\mathbf{0}_i] = [(0, i)]$, $i \in I$, for the unique vertex of the star graph.[5]

2 Throughout this thesis, we denote by I the set of indices $\{1, \dots, m\}$, where $m \in \mathbb{N}$.
3 Schematic representation; rays are indistinguishable with no associated angles.
4 Throughout this thesis, bold symbols denote points, functions, and operators on $\boldsymbol{\mathcal{S}_m}$.
5 Here, $[\cdot]$ denotes the equivalence class under the relation $\sim$.

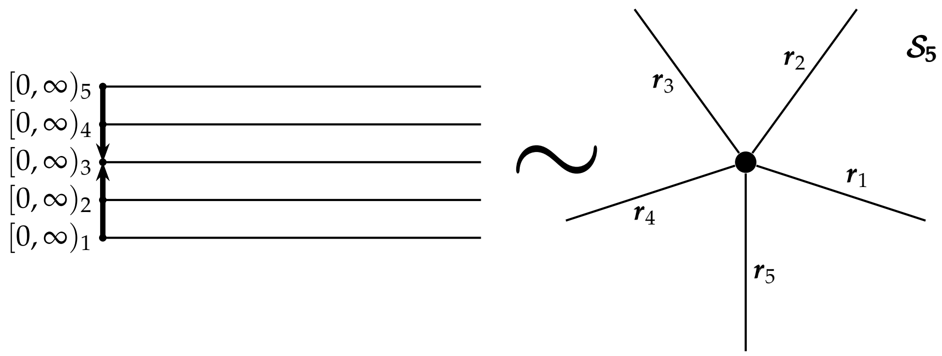

Figure 2.2: Construction of the metric star graph $\boldsymbol{\mathcal{S}_5}$.[3]

Since all the rays in the star graph $\boldsymbol{\mathcal{S}_m}$ are indistinguishable, this star graph is invariant by any permutation of the rays.

Finally, by following [Mug21], we endow $\boldsymbol{\mathcal{S}_m}$ with the path metric $d_{\boldsymbol{\mathcal{S}_m}}$ defined as

$$d_{\boldsymbol{\mathcal{S}_m}}(\xi,\theta) := \inf \sum_{l=1}^{k} d_{S_m}(\xi_l,\theta_l), \quad \xi,\theta \in \boldsymbol{\mathcal{S}_m}, \tag{2.15}$$

where the infimum is taken over all $k \in \mathbb{N}$ and all pairs of k-tuples $(\xi_1,\dots,\xi_k)$ and $(\theta_1,\dots,\theta_k)$ where $\xi_l,\theta_l \in S_m$ for all $l \in \{1,\dots,k\}$, with $\xi_1 \sim \xi, \theta_k \sim \theta$, and $\theta_i \sim \xi_{i+1}$ for all $i \in \{1,\dots,k-1\}$. For arbitrary quotient spaces, the definition (2.15) yields a pseudo metric. However, in our scenario, characterized by a finite number of edges, this pseudo-metric manifests itself as a metric. Moreover, it is a metric in the strict sense since the value ∞ is unattainable as our underlying graph is connected. A closer examination reveals that this path metric is also known as the SNCF or Paris metric. In other words:

$$d_{\boldsymbol{\mathcal{S}_m}}\big((x,i),(y,j)\big) = \begin{cases} |x|+|y|, \text{if } i \neq j, \\ |x-y|, \quad \text{otherwise.} \end{cases}$$

A basis for the topology induced by $d_{\boldsymbol{\mathcal{S}_m}}$ consists of open balls with respect to $d_{\boldsymbol{\mathcal{S}_m}}$. These open balls are either open subintervals of the rays or, up to glueing with $\sim$, disjoint unions of semi-open and equal-length subintervals.

The disjoint union S_m is also a measure space with the sigma-algebra Σ of the disjoint union and the direct sum measure $\gamma := \bigoplus_{i\in I} \gamma_i$, cf. [Fre11, 214K].

Via the canonical surjection $q\colon S_m \to \boldsymbol{\mathcal{S}_m}$, we construct $(\boldsymbol{\mathcal{S}_m},\boldsymbol{\Sigma},\boldsymbol{\gamma})$, the quotient measure space induced by $\sim$ on S_m.. Following [Bog06, Section 3.6] we define the pushforward σ-algebra $\boldsymbol{\Sigma}$ on $\boldsymbol{\mathcal{S}_m}$ as the collection of all sets $\boldsymbol{E} \subset \boldsymbol{\mathcal{S}_m}$ such that $q^{-1}(\boldsymbol{E}) \in \Sigma$. This ensures that q is measurable and compatible with Σ, respecting the equivalence relation. We define the measure $\boldsymbol{\gamma}$ on $\boldsymbol{\Sigma}$ by the pushforward formula:

$$\boldsymbol{\gamma}(\boldsymbol{E}) = \gamma(q^{-1}(\boldsymbol{E})) \quad \text{for all} \quad \boldsymbol{E} \in \boldsymbol{\Sigma}. \tag{2.16}$$

Remark 2.10 *The pushforward measure $\boldsymbol{\gamma}$ is introduced to ensure compatibility with the equivalence relation $\sim$ on the disjoint union S_m. While the absolute continuity of γ with respect to the Lebesgue measure implies that singleton sets are null sets, the pushforward construction ensures that the measure $\boldsymbol{\gamma}$ is well-defined on the quotient space $\boldsymbol{\mathcal{S}_m} = S_m/\sim$.*

This construction may not be strictly necessary for the subsequent theory developed in the work, but it provides a rigorous foundation for the measure-theoretic aspects of the problem.

Remark 2.11 *To ensure the well-definedness of the pushforward measure $\boldsymbol{\gamma}$ on the quotient space $\boldsymbol{\mathcal{S}_m}$, we note that for any $E \in \Sigma$,*

$$\gamma(q^{-1}(q(E)) \setminus E) = 0. \tag{2.17}$$

This condition guarantees that the measure of the preimage of a set under the canonical surjection q differs from the measure of the original set by at most a set of measure zero.

Remark 2.12 *The construction of measures on metric star graphs presented in this thesis can be extended to any finite metric graph, including those with infinite leads. For a more detailed discussion on the construction of measures on general metric graphs, we refer the reader to [Mug21].*

Following the change of variables formula [Bog06, Theorem 3.6.1] and the integration on the disjoint union [Fre11, p. 214M] the integration on $\boldsymbol{\mathcal{S}_m}$ for a positive and measurable function $\boldsymbol{f}\colon \boldsymbol{\mathcal{S}_m} \to \mathbb{R}$ is given as:

$$\begin{aligned}\int_{\boldsymbol{\mathcal{S}_m}} \boldsymbol{f}(\boldsymbol{x})\boldsymbol{\gamma}(\mathrm{d}\boldsymbol{x}) = \int_{S_m} \boldsymbol{f}\big(q(x)\big)\gamma(\mathrm{d}x) &= \sum_{i\in I}\int_0^\infty f_i(x)\gamma_i(\mathrm{d}x) \\ &= \frac{2}{m\sqrt{\pi}}\sum_{i\in I}\int_0^\infty f_i(x)e^{-x^2}\,\mathrm{d}x.\end{aligned}$$

where $f_i\colon [0,\infty) \to \mathbb{R}$, $f_i : x \mapsto \boldsymbol{f}\big(q(\phi_i(x))\big)$.

Note that $\boldsymbol{f}$ is measurable if and only if f_i is measurable for every $i \in I$, [Fre11, p. 214M].

This result can be extended to real or complex functions $\boldsymbol{f}\colon \boldsymbol{\mathcal{S}_m} \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, as long as $\int_0^\infty |f_i(x)|\gamma_i(dx) < \infty$ for all $i \in I$.

In this sense, every measurable function $\boldsymbol{f}\colon \boldsymbol{\mathcal{S}_m} \to \mathbb{K}$ can be equivalently regarded as a family $(f_i)_{i\in I}$ of measurable functions $f_i\colon [0,\infty) \to \mathbb{K}$. Furthermore, let $C(\boldsymbol{\mathcal{S}_m})$ denote the space of continuous functions on $\boldsymbol{\mathcal{S}_m}$, i.e., the space of all functions $f \in C(\boldsymbol{\mathcal{S}_m})$ corresponding to a family $(f_i)_{i\in I}$ of continuous functions $f_i\colon [0,\infty) \to \mathbb{K}$ satisfying the compatibility condition $f_i(0) = f_j(0)$ for all $i,j \in I$. This common value at the origin is denoted by $f(\mathbf{0})$, i.e., $f(\mathbf{0})\colon = f_i(0)$ for any $i \in I$.

Therefore, $\boldsymbol{\mathcal{S}_m}$ is equipped with a path metric and a measure $\boldsymbol{\gamma}$ defined by the pushforward of the direct sum measure constructed from the Gaussian measures γ_i defined on each ray $\boldsymbol{r}_i$. This framework allows us to introduce the function spaces $C_b(\boldsymbol{\mathcal{S}_m})$ of bounded and continuous functions on $\boldsymbol{\mathcal{S}_m}$ and the Gaussian space $L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m}), p \in [1,\infty)$.

We now introduce several key concepts and function spaces that form the foundation for our analysis on the metric star graph $\boldsymbol{\mathcal{S}_m}$. We begin with the Gaussian Lebesgue space and its associated scalar product and norm.

Definition 2.13 ($L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$ Gaussian Lebesgue Space) *The Gaussian Lebesgue space $L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$ is defined as the space of $\boldsymbol{\gamma}$-measurable functions $\boldsymbol{f}: \boldsymbol{\mathcal{S}_m} \to \mathbb{K}$ for which the following norm is finite:*

$$\|\boldsymbol{f}\|_{L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})} := \left(\sum_{i \in I} \int_0^\infty |f_i(x)|^2 \, \gamma_i(\mathrm{d}x) \right)^{1/2} < \infty, \tag{2.18}$$

where f_i denotes the restriction of $\boldsymbol{f}$ to the ray $\boldsymbol{r_i}$ of the star graph.

Definition 2.14 (Scalar Product on $L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$) *For $\boldsymbol{f}, \boldsymbol{g} \in L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$, the scalar product $\langle \cdot, \cdot \rangle_{L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})}$ is defined as*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})} := \int_{\boldsymbol{\mathcal{S}_m}} \boldsymbol{f}(\boldsymbol{x}) \overline{\boldsymbol{g}(\boldsymbol{x})} \, \boldsymbol{\gamma}(\mathrm{d}\boldsymbol{x}) = \sum_{i \in I} \int_0^\infty f_i(x) \overline{g_i(x)} \, \gamma_i(\mathrm{d}x), \tag{2.19}$$

where g_i denotes the restriction of $\boldsymbol{g}$ to the ray $\boldsymbol{r_i}$.

Remark 2.15 *The norm $\| \cdot \|_{L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})}$ is induced by the scalar product:*

$$\|\boldsymbol{f}\|^2_{L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})} = \langle \boldsymbol{f}, \boldsymbol{f} \rangle_{L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})}. \tag{2.20}$$

We now generalize this concept to define Gaussian Lebesgue spaces for other p values and introduce Gaussian Sobolev spaces.

Definition 2.16 ($L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$ Gaussian Lebesgue Space) *For $1 \leq p < \infty$, the Gaussian Lebesgue space $L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$ is defined as the space of $\boldsymbol{\gamma}$-measurable functions $\boldsymbol{f}: \boldsymbol{\mathcal{S}_m} \to \mathbb{K}$ for which*

$$\|\boldsymbol{f}\|_{L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})} := \left(\int_{\boldsymbol{\mathcal{S}_m}} |\boldsymbol{f}(\boldsymbol{x})|^p \, \boldsymbol{\gamma}(\mathrm{d}\boldsymbol{x}) \right)^{1/p} < \infty.$$

Definition 2.17 ($W^{k,p}_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$ Gaussian Sobolev Space) *For $k \in \mathbb{N}_0$ and $1 \leq p < \infty$, the Gaussian Sobolev space $W^{k,p}_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$ is defined as*

$$W^{k,p}_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m}) := \left\{ \boldsymbol{f} \in L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m}) : D^\alpha \boldsymbol{f} \in L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m}), |\alpha| \leq k \right\},$$

where $W^{0,p}_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m}) := L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})$, α is a multi-index, and $D^\alpha \boldsymbol{f}$ is understood as a weak derivative of $\boldsymbol{f}$. We endow this space with the norm

$$\|\boldsymbol{f}\|_{W^{k,p}_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})} := \left(\sum_{|\alpha| \leq k} \|D^\alpha \boldsymbol{f}\|^p_{L^p_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m})} \right)^{1/p}.$$

These function spaces and the scalar product provide the framework for studying differential operators and their properties on $\boldsymbol{\mathcal{S}_m}$. In particular, they allow us to define the minimal and maximal operators associated with $\mathcal{A}$ and to investigate the vertex conditions that characterize the self-adjoint extensions of the minimal operator.

2.3 VERTEX CONDITIONS $\mathcal{C}(\mathbf{0})$

To complete the characterization of the quantum graph triple $G_{OU} = (\boldsymbol{\mathcal{S}_m}, \boldsymbol{\mathcal{A}}, \mathcal{C}(\mathbf{0}))$, we now focus on the conditions $\mathcal{C}(\mathbf{0})$ on the central vertex $\mathbf{0}$. These conditions are key in determining the self-adjoint extensions of $\boldsymbol{\mathcal{A}}$ on $\boldsymbol{\mathcal{S}_m}$. We begin by introducing the minimal and maximal operators associated with $\boldsymbol{\mathcal{A}}$.

Let $\boldsymbol{\mathcal{A}}$ be the OU differential expression on the metric star graph $\boldsymbol{\mathcal{S}_m}$, given by

$$(\boldsymbol{\mathcal{A}} \boldsymbol{f})(\boldsymbol{x}_i) = \frac{1}{2} \boldsymbol{f}''(\boldsymbol{x}_i) - |\boldsymbol{x}_i| \boldsymbol{f}'(\boldsymbol{x}_i), \quad \boldsymbol{x}_i \in \boldsymbol{\mathcal{S}_m}, i \in I. \tag{2.21}$$

The minimal operator $\boldsymbol{A}^{\min}$ is defined as the closure—with respect to the norm (2.20)—of the operator $\boldsymbol{\mathcal{A}}$ on the domain

$$\mathrm{Dom}(\boldsymbol{A}^{\min}) := C_0^\infty(\boldsymbol{\mathcal{S}_m} \setminus \{\boldsymbol{0}\}), \tag{2.22}$$

which consists of smooth functions with compact support separated from the central vertex $\boldsymbol{0}$. This operator is symmetric, but not self-adjoint, in the Hilbert space $L^2_\gamma(\boldsymbol{\mathcal{S}_m})$.

On the other hand, the maximal operator $\boldsymbol{A}^{\max}$ is defined on the domain

$$\mathrm{Dom}(\boldsymbol{A}^{\max}) := \left\{ \boldsymbol{f} \in L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m}) : \boldsymbol{\mathcal{A}} \boldsymbol{f} \in L^2_{\boldsymbol{\gamma}}(\boldsymbol{\mathcal{S}_m}) \right\}. \tag{2.23}$$

The self-adjoint extensions of $\boldsymbol{A}^{\min}$ are precisely the restrictions of $\boldsymbol{A}^{\max}$ to domains satisfying specific conditions at the central vertex $\boldsymbol{0}$. These VCs are characterized by the maximal operator's inherent *boundary form*, defined as follows:

Definition 2.18 (Boundary Form) *The boundary form associated with the maximal operator $\boldsymbol{A}^{\max}$ is the sesquilinear form defined on $D(\boldsymbol{A}^{\max}) \times D(\boldsymbol{A}^{\max})$ by*

$$\langle \boldsymbol{A}^{\max} \boldsymbol{u}, \boldsymbol{v} \rangle_{L^2_\gamma(\boldsymbol{\mathcal{S}_m})} - \langle \boldsymbol{u}, \boldsymbol{A}^{\max} \boldsymbol{v} \rangle_{L^2_\gamma(\boldsymbol{\mathcal{S}_m})} = \langle \partial \boldsymbol{u}(\boldsymbol{0}), \boldsymbol{v}(\boldsymbol{0}) \rangle_{\mathbb{K}^m} - \langle \boldsymbol{u}(\boldsymbol{0}), \partial \boldsymbol{v}(\boldsymbol{0}) \rangle_{\mathbb{K}^m}, \tag{2.24}$$

where $\boldsymbol{u}(\boldsymbol{0}) := \big(\boldsymbol{u}(0_1), \ldots, \boldsymbol{u}(0_m)\big)$[6]*, $\partial \boldsymbol{u}(\boldsymbol{0}) := \big(\boldsymbol{u}'(0_1), \ldots, \boldsymbol{u}'(0_m)\big)$, and $\langle \cdot, \cdot \rangle_{\mathbb{K}^m}$ denotes the standard scalar product in $\mathbb{K}^m$ scaled by the factor $\frac{2}{m\sqrt{\pi}}$ arising from the measure γ.*

To characterize the self-adjoint extensions of $\boldsymbol{A}^{\min}$, we employ the concept of *Hermitian VCs*, as introduced by [Kur23].

Definition 2.19 (Hermitian VCs) [Kur23, Definition 3.1] *Let V be the space of limit values $(\boldsymbol{u}, \partial \boldsymbol{u}) \in \mathbb{K}^{2m}$ at a vertex of degree m. A subspace $V_C \subseteq V$ defines* Hermitian *VCs if and only if it satisfies the following:*

- *The boundary form* (2.24) *vanishes for all $(\boldsymbol{u}, \partial \boldsymbol{u}), (\boldsymbol{v}, \partial \boldsymbol{v}) \in V_C$.*
- *The subspace V_C has dimension m.*

The following theorem characterizes Hermitian VCs in terms of matrices B and C satisfying certain rank and Hermiticity conditions.

Theorem 2.20 [Kur23, Theorem 3.2] *Let $\boldsymbol{0}$ be a vertex of degree m in a quantum graph. Any Hermitian VC at $\boldsymbol{0}$ admits a representation of the form*

$$\boldsymbol{B} \boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{C} \partial \boldsymbol{u}(\boldsymbol{0}), \tag{2.25}$$

where

6 Recall that we denote points $\boldsymbol{x}_i$ in $\boldsymbol{\mathcal{S}_m}$ as $\boldsymbol{x}_i = (x, i)$, where $x \in [0, \infty)$ and $i \in I$.

- $\boldsymbol{u}(\mathbf{0})$ *and* $\partial\boldsymbol{u}(\mathbf{0})$ *are the* $\mathbb{K}^m$*-vectors of function and derivative values at the vertex.*
- $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are* $m \times m$ *matrices satisfying*

$$\operatorname{rank}(\boldsymbol{B},\boldsymbol{C}) = m \quad \textit{and} \quad \boldsymbol{BC}^* \textit{ is Hermitian}^7. \tag{2.26}$$

Remark 2.21 *Although theorem 2.20 is originally stated for Schrödinger operators in [Kur23], the proof remains valid for the* $\boldsymbol{\mathcal{A}}$ *since the boundary forms for both operators coincide with* (2.24) *up to a real constant.*

Theorem 2.20 establishes a one-to-one correspondence between the self-adjoint extensions of $\boldsymbol{A}^{\min}$ and the Hermitian conditions on the central vertex $\mathbf{0}$. More precisely, every self-adjoint extension of $\boldsymbol{A}^{\min}$ can be uniquely determined by specifying the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ satisfying the conditions (2.26). However, it is important to note that the choice of matrices B and C for a given self-adjoint extension is not unique, as any pair of matrices $(\boldsymbol{MB},\boldsymbol{MC})$, where $\boldsymbol{M}$ is a non-singular $m \times m$ matrix, leads to the same boundary condition as $(\boldsymbol{B},\boldsymbol{C})$.

Example 2.22 *The* Standard VCs, *mandating continuity and adherence to the Kirchhoff condition, represent a notable instance of Hermitian VCs.*

$$\boldsymbol{u}(\mathbf{0}_k) = \boldsymbol{u}(\mathbf{0}_l) \quad \textit{and} \quad \sum_{i\in I} \boldsymbol{u}'(\mathbf{0}_i) = 0 \quad \textit{for all } k,l \in I. \tag{2.27}$$

These conditions ensure the self-adjointness of the OU operator and reflect the natural connectivity of the metric star graph $\boldsymbol{\mathcal{S}_m}$.

For the Standard VCs, the matrices $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *can be chosen as follows:*

$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{2.28}$$

It can be easily verified that the rank of $(\boldsymbol{B},\boldsymbol{C})$ *is* m *and that* $\boldsymbol{BC}^* = \boldsymbol{CB}^* = 0$, *satisfying the conditions of Theorem 2.20.*

In summary, Theorem 2.20 provides a characterization of the VCs $\mathcal{C}(\mathbf{0})$ for $\boldsymbol{\mathcal{A}}$ on $\boldsymbol{\mathcal{S}_m}$. The Standard VCs, encompassing both continuity and the Kirchhoff condition, serve as a canonical example of such Hermitian VCs. With this characterization, we have completed the definition of the quantum graph triple $G_{OU} = (\boldsymbol{\mathcal{S}_m},\boldsymbol{\mathcal{A}},\mathcal{C}(\mathbf{0}))$, thus providing the essential framework for the investigation of the existence and properties of the associated operator and semigroups.

7 A matrix M is Hermitian if it is equal to its conjugate transpose, i.e., $M = M^*$. In our case, this translates to the condition $\boldsymbol{BC}^* = \boldsymbol{CB}^*$.

3 EXISTENCE AND PROPERTIES OF ORNSTEIN-UHLENBECK SEMIGROUPS ON METRIC STAR GRAPHS

Qual è 'l geomètra che tutto s'affige per misurar lo cerchio,
e non ritrova, pensando, quel principio ond' elli indige; [1]

— Dante Alighieri [Ali08]

3.1 INTRODUCTION

In recent years, there has been a growing interest in extending the theory of Ornstein-Uhlenbeck (OU) operators to more general settings, such as metric graphs. Mugnolo and Rhandi [MR22] make significant contributions to this area by investigating unbounded-drift diffusion processes on metric star graphs, with a particular focus on the OU operator. Their work extends the previously known theory in several key aspects.

First and foremost, they investigate the well-posedness of parabolic equations on metric star graphs, specifically in the classical sense and subject to Kirchhoff Vertex Conditions (VCs). This approach had not been previously addressed in the context of OU operators on metric graphs. The central vertex, being a single point connecting all branches, presents a challenge in the analysis of such problems, as the Standard VCs (see 2.22) imposed at this vertex resemble Boundary Conditions (BCs) despite it being an interior point of the star graph.

To overcome this challenge, they employ a technique that combines the use of truncated star graphs and *the method of even-odd extensions* (see Lemma 3.4). They consider truncated star graphs $\mathcal{S}_m^n$, obtained by restricting each half-line of the original star graph $\mathcal{S}_m$ to the interval $[0, n)$. By applying the classical theory of parabolic equations, which guarantees the existence and uniqueness of solutions on bounded domains under appropriate BCs, to these truncated graphs and employing Schauder estimates, they construct solutions on the entire star graph $\mathcal{S}_m$.

Moreover, they introduce an associated operator $\widetilde{L}$ on the real line, extending not only the domain but also the operator beyond the star graph. This extension presents an additional challenge: to apply the method of even-odd extensions effectively, the extended operator must be invariant under the transformation $x \mapsto -x$. Mugnolo and Rhandi construct $\widetilde{L}$ to preserve this symmetry, defining its coefficients as even or odd extensions of those of L. The symmetry of the extended operator $\widetilde{L}$ allows them to take advantage of a classical key result. If the initial data is even/odd, then the solution will inherit the same parity with respect to the central vertex. This parity preservation property is essential in verifying that the patched solutions $\hat{u}_n$ on the truncated star graphs satisfy the Standard VCs at the central vertex. This approach, combined with the use of truncated star graphs,

1 Like the geometer, who gives himself wholly to measuring the circle,
nor, by thinking, finds the principle he needs; [Ali21]

allows them to demonstrate the well-posedness of parabolic problems on metric star graphs, explicitly establishing the existence of a unique classical solution, as stated in [MR22, Theorem 3.3].

In addition to the existence and uniqueness results, they provide a closed-form expression for the OU semigroup on metric star graphs, generalizing the well-known results for the OU semigroup on $\mathbb{R}$. They also investigate the properties of this semigroup, such as its invariant measure and long-time behavior, which are relevant for understanding the dynamics of the OU process on metric star graphs.

The following sections will analyze the main results presented in [MR22]. Section 3.2 focuses on [MR22, Theorem 3.3], which establishes the existence of classical solutions to parabolic equations on metric star graphs with unbounded coefficients. In Section 3.3, the close-form expression for the OU semigroup and its properties are discussed in detail. Section 3.4 explores the spectral properties and long-time behavior of the OU operator on metric star graphs, providing insights into the dynamics of the associated process. The chapter concludes with an outlook on future directions and potential extensions of the presented results.

3.2 EXISTENCE OF CLASSICAL SOLUTIONS

In this section, we focus on the central problem addressed in [MR22], namely the existence of classical solutions to parabolic problems on metric star graphs. We begin by setting up the necessary framework and introducing the key concepts and definitions.

Considering the space $C_b(\boldsymbol{\mathcal{S}_m})$ consisting of functions on the metric star graph $\boldsymbol{\mathcal{S}_m}$ (Section 2.2) that are both continuous and bounded, [MR22] study the Kolmogorov differential operator $\boldsymbol{\mathcal{L}}$ defined by[2]:

$$\boldsymbol{\mathcal{L}} \boldsymbol{f}(\boldsymbol{x}_i) = q(|\boldsymbol{x}_i|)\boldsymbol{f}''(\boldsymbol{x}_i) + b(|\boldsymbol{x}_i|)\boldsymbol{f}'(\boldsymbol{x}_i) + c(|\boldsymbol{x}_i|)\boldsymbol{f}(\boldsymbol{x}_i), \quad i \in I,$$

where q, b, and c are continuous functions satisfying the following conditions:

- $q, b, c \in C^{\nu}_{\text{loc}}([0, \infty))$ for $\nu \in (0, 1)$,
- $b(0) = 0$,
- $q(x) > 0$, $\forall x \in [0, \infty)$,
- $\sup c \leq c_0$, for $c_0 \in \mathbb{R}$.

where $C^{\nu}_{\text{loc}}([0, \infty))$ denotes the space of locally Hölder continuous functions with exponent ν, i.e., functions g such that for every interval $[a, b] \subset [0, \infty)$, there exists a constant $C_K > 0$ satisfying

$$|g(x) - g(y)| \leq C_K |x - y|^{\nu} \quad \text{for all } x, y \in [a, b].$$

2 Throughout this thesis, bold symbols (e.g., $\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{A}$) are used to denote points in the star graph, functions on the star graph, and operators acting on functions on the star graph, respectively.

The Kolmogorov differential expression $\boldsymbol{\mathcal{L}}$ gives rise to an operator $\boldsymbol{L}$ when its domain is specified. The domain of $\boldsymbol{L}$ incorporates the Kirchhoff VCs at the central vertex by requiring functions to satisfy $\sum_{i\in I} f_i'(0) = 0$:[3]

$$\mathrm{Dom}(\boldsymbol{L}) = \left\{ \boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m}) \cap \left(\bigcap_{1\le p<\infty} \widetilde{W^{k,p}_{\mathrm{loc}}}(\boldsymbol{\mathcal{S}_m}) \right) \ : \ \sum_{i\in I} f_i'(0) = 0 \text{ and } \boldsymbol{\mathcal{L}f} \in C_b(\boldsymbol{\mathcal{S}_m}) \right\},$$

where $\widetilde{W^{k,p}_{\mathrm{loc}}}(\boldsymbol{\mathcal{S}_m}) := \bigoplus_{i=1}^{m} W^{k,p}_{\mathrm{loc}}([0,\infty))$, and $W^{k,p}_{\mathrm{loc}}([0,\infty))$ denotes the space of functions g such that for every closed interval $[a,b] \subset [0,\infty)$, the restriction $g|_{[a,b]}$ belongs to the Sobolev space $W^{k,p}\big([a,b]\big)$.

To study the time-dependent behavior associated with the operator $\boldsymbol{L}$, ([MR22]) consider the corresponding *parabolic problem*, which describes the evolution of a function $\boldsymbol{u}(t,\cdot)$ over time, subject to the operator $\boldsymbol{L}$ and an initial condition g.

To formalize this notion, the authors introduce the following definitions:

Definition 3.1 (Parabolic Problem) *The parabolic problem for a generic operator* Λ *is given by*

$$\begin{cases} \partial_t u(t,\cdot) = \Lambda u(t,\cdot), & t>0, \\ u(0,\cdot) = f(\cdot), \end{cases} \tag{P_Λ}$$

where $u \in \mathrm{Dom}(\Lambda)$ *and* f *is a given initial condition.*

Definition 3.2 (Classical Solution) *We define a* classical solution *of the parabolic problem* $(P_{\boldsymbol{L}})$ *to be a function* $\boldsymbol{u} \in C_b\big([0,\infty) \times \boldsymbol{\mathcal{S}_m}\big)$ *such that:*

- $\boldsymbol{u}(\cdot,\boldsymbol{x}) \in C^1\big((0,\infty)\big)$ *for every* $\boldsymbol{x} \in \boldsymbol{\mathcal{S}_m}$*,*
- $\boldsymbol{u}(t,\cdot) \in \mathrm{Dom}(\boldsymbol{L})$ *for every* $t>0$*,*
- $\boldsymbol{u}$ *satisfies* $(P_{\boldsymbol{L}})$*.*

The central theorem in [MR22] establishes the existence of classical solutions for $(P_{\boldsymbol{L}})$. While the classical theory of parabolic equations ensures the existence and uniqueness of solutions under suitable BCs, it cannot be directly applied to the star graph $\boldsymbol{\mathcal{S}_m}$ due to the Standard VCs imposed at the central vertex.

To circumvent this obstacle, one can employ the method of even-odd extensions. Extending the problem to a larger domain and constructing an associated operator $\widetilde{L}$ on the real line that extends the original operator $\boldsymbol{L}$ beyond the star graph.

The associated operator $\widetilde{L}$ acting on $C_b(\mathbb{R})$ is defined as

$$\widetilde{L}g(x) = \widetilde{q}(x)g''(x) + \widetilde{b}(x)g'(x) + \widetilde{c}(x)g(x), \quad x \in \mathbb{R},$$

with domain

$$\mathrm{Dom}(\widetilde{L}) = \left\{ g \in C_b(\mathbb{R}) \cap \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}) : \widetilde{L}g \in C_b(\mathbb{R}) \right\},$$

3 [MR22] are effectively studying the quantum graph $G_K := \big(\boldsymbol{\mathcal{S}_m}, \boldsymbol{\mathcal{L}}, \mathcal{C}(\mathbf{0})\big)$, where $\mathcal{C}(\mathbf{0})$ represents the Standard VCs imposed at the central vertex $\mathbf{0}$.

where the coefficients $\widetilde{q}$, $\widetilde{b}$, and $\widetilde{c}$ are defined as

$$\widetilde{q}(y) = \begin{cases} q(y), & y \geq 0, \\ q(-y), & y \leq 0, \end{cases} \quad \widetilde{b}(y) = \begin{cases} b(y), & y \geq 0, \\ -b(-y), & y \leq 0, \end{cases} \quad \widetilde{c}(y) = \begin{cases} c(y), & y \geq 0, \\ c(-y), & y \leq 0. \end{cases}$$

Even extension Odd extension Even extension

The key property of $\widetilde{L}$ is its invariance under the transformation $x \mapsto -x$, meaning that this equality $(\widetilde{L}f)(x) = (\widetilde{L}f)(-x)$ is valid for all functions f in its domain. This invariance, together with the classical theory of partial differential equations, implies that if the initial condition f is odd (resp. even), then the solution $u(x,t)$ to the parabolic problem associated with $\widetilde{L}$ is odd (resp. even) in x for all $t > 0$. These parity properties ensure that the extended solution satisfies the desired BCs when restricted to the original star graph.

With the associated operator $\widetilde{L}$ and the parity properties of the solutions at hand, the authors proved the following theorem:

Theorem 3.3 [MR22, Theorem 3.3] *Let q, b and c be functions in $C^{\nu}_{\text{loc}}([0,\infty))$, where $\nu \in (0,1)$, satisfying the conditions $q(x) > 0$ for all $x \in [0,\infty)$, $\sup c \leq c_0$ for $c_0 \in \mathbb{R}$ and $b(0) = 0$. Under these assumptions, for every function $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$, the parabolic problem $(P_{\boldsymbol{L}})$ admits at least one classical solution.*

Moreover, the uniqueness of the solution to $(P_{\widetilde{L}})$ guarantees the uniqueness of the solution of $(P_{\boldsymbol{L}})$. In this scenario the semigroup $(\boldsymbol{T_m}(t))_{t\geq 0}$ generated by $\boldsymbol{L}$ on $C_b(\boldsymbol{\mathcal{S}_m})$ is represented by

$$\big(\boldsymbol{T_m}(t)\boldsymbol{f}\big)(\boldsymbol{x}_i) = \int_{\boldsymbol{r_i}} (\kappa(t, |\boldsymbol{x}_i|, |\boldsymbol{y}_i|) - \kappa(t, |\boldsymbol{x}_i|, -|\boldsymbol{y}_i|)) f(\boldsymbol{y}_i) \mathrm{d}\boldsymbol{y}_i \tag{3.1}$$

$$+ \sum_{j\in I} \int_{\boldsymbol{r_j}} \frac{2}{m} \kappa(t, |\boldsymbol{x}_i|, -|\boldsymbol{y}_j|) f(\boldsymbol{y}_j) \mathrm{d}\boldsymbol{y}_j. \tag{3.2}$$

where $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$, $\boldsymbol{r_i}$ denotes the i-th ray[4], $\boldsymbol{x_i} \in \boldsymbol{\mathcal{S}_m}$ for $i \in I$, and with integral kernel κ.[5] Additionally, under the conditions $c \equiv 0$ and $T(t_0)\mathbb{1} = \mathbb{1}$ for some $t_0 > 0$, we have $\boldsymbol{T_m}(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$, implying the conservativeness of the semigroup $\boldsymbol{T_m}(\cdot)$.

The proof of Theorem 3.3 relies on the general existence result for parabolic problems, [LB06, Theorem 2.2.5]. By extending the problem to the real line and drawing upon the symmetry properties of the metric star graph, [MR22] construct solutions on the truncated star graphs that satisfy the necessary Standard VCs at the central vertex. The existence of a solution on the full star graph is then obtained by passing to the limit. [6]

Before proceeding with the proof of Theorem 3.3, we establish a key lemma that formalizes the method of even-odd extensions employed by [MR22].

4 See Definition (2.8)

5 Here, κ represents the integral kernel associated with the strongly continuous semigroup $(T(t))_{t\geq 0}$ generated by the operator $\widetilde{L}$.

6 This limiting procedure relies on Schauder estimates to establish uniform bounds on derivatives of the solutions on truncated domains, and the Arzelà–Ascoli theorem to extract a convergent subsequence.

Lemma 3.4 (Even-Odd Extension) *Consider a metric star graph $\boldsymbol{\mathcal{S}_m}$ with $m \in \mathbb{N}$ rays. For each non-zero $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$, satisfying $\sum_{i\in I} f_i'(0) = 0$,*[7]*, we define the extended functions $\widetilde{f}_i \in C_b(\mathbb{R})$, $i \in I$, as follows:*

$$\widetilde{f}_i(x) := \begin{cases} f_i(x), & \text{if } x \geq 0, \\ -f_i(-x) + \dfrac{2}{m} \sum_{j\in I} f_j(-x), & \text{if } x \leq 0. \end{cases} \tag{3.3}$$

Then, the functions $\widetilde{f}_i$ satisfy the following properties:

(i) $\widetilde{f}_i$ agrees with $\boldsymbol{f}$ on the i-th ray of $\boldsymbol{\mathcal{S}_m}$,

(ii) the sum $\sum_{i\in I} \widetilde{f}_i$ is an even function, and

(iii) the sum $\sum_{i\in I} \widetilde{f}_i'$ is an odd function, thus satisfying the Kirchhoff VC $\sum_{i\in I} \widetilde{f}_i'(0) = 0$.

(iv) the functions $\widetilde{f}_{ij}(x) := \widetilde{f}_i(x) - \widetilde{f}_j(-x)$ are odd for all $i, j \in I$.

The domain extension is illustrated in Figure 3.1. The figure on the left-hand side shows the original star graph $\boldsymbol{\mathcal{S}_m}$ with rays $\boldsymbol{r}_i$, while the image on the right depicts the extended graph obtained by extending the domain, introducing the negative half-lines as the domains for the extended functions $\widetilde{f}_i$ and $\widetilde{f}_{i,j}$.

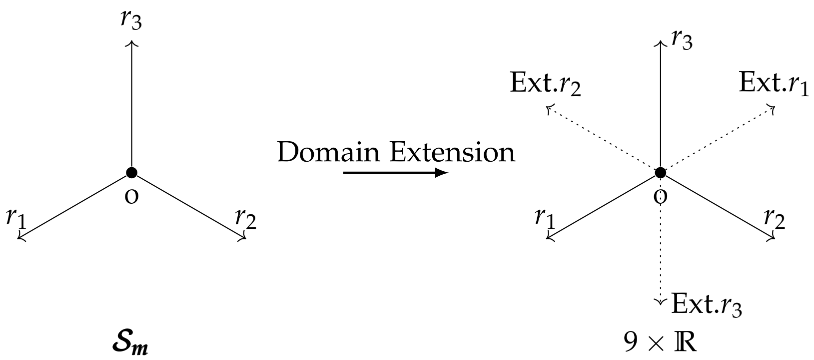

Figure 3.1: Domain extension of the metric star graph $\boldsymbol{\mathcal{S}_m}$.

Proof Given $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$ satisfying the Kirchhoff VC, we construct a family of functions $\widetilde{f}_i \in C_b(\mathbb{R})$, $i \in I$, as follows:

$$\widetilde{f}_i(x) := \begin{cases} f_i(x), & \text{if } x \geq 0, \\ g_i(x), & \text{if } x \leq 0, \end{cases} \tag{3.4}$$

where g_i is to be determined properly.

By construction, $\widetilde{f}_i$ coincides with f_i on the i-th ray of $\boldsymbol{\mathcal{S}_m}$ thus establishing property (i).

We assume now that the extension to the negative axis, g_i, is constructed by linearly combining the information available from all the m-rays, i. e., by expressing $g_i(x)$ as a linear combination of $f_j(-x)$:

7 Recall that $\boldsymbol{f} \equiv (f_i)_{i\in I}$ for $f_i \in C_b([0,\infty])$ and $f_i(0) = \boldsymbol{f}(\boldsymbol{0})$ for all $i \in I$.

$$g_i(x) := \sum_{j\in I} w_{ij} f_j(-x), \quad x \in (-\infty, 0], \tag{3.5}$$

where the *weights* $w_{ij} \in \mathbb{K}$ quantify the contribution of f_j to the extension $g_i|_{(-\infty,0]}$ of f_i.

To ensure continuity at $x = 0$, we require $g_i(0) = f_i(0) = \boldsymbol{f}(\mathbf{0})$ for all $i \in I$, which leads to the condition

$$\sum_{j\in I} w_{ij} = 1. \tag{3.6}$$

To achieve the sum-even extension, we impose the following symmetry-motivated conditions on the matrix $W := (w_{ij})_{i,j\in I}$:

- W is symmetric, i.e., $W = W^T$ (The contributions between rays i and j are reciprocal.
- All off-diagonal entries of W are equal to some constant $p \in \mathbb{K}$ (All rays contribute equally to the extensions on other axes).
- All diagonal entries of W are equal. (The self-contribution of each ray to its own extension is the same.

From (3.6), it follows that the diagonal elements are given by $w_{ii} = 1 - p(m-1)$. A direct calculation shows that for $x \le 0$:

$$\sum_{i\in I} g_i(x) = \sum_{i\in I}\left(\sum_{j\in I} w_{ij} f_j(-x)\right) = \sum_{j\in I} f_j(-x) \sum_{i\in I} w_{ji} = \sum_{j\in I} f_j(-x), \tag{3.7}$$

implying that the sum function $\mathbb{R} \ni x \mapsto \sum_{i\in I} \widetilde{f}_i(x)$ is an even function (see property (ii)). Consequently, its derivative $\mathbb{R} \ni x \mapsto \sum_{i\in I} \widetilde{f}_i'(x)$ is an odd function since the hypothesis $\sum_{i\in I} f_i'(0) = 0$ ensures the continuity of $\sum_{i\in I} \widetilde{f}_i'(x)$ at $x = 0$ and guarantees that it fulfills the Kirchhoff VC $\sum_{i\in I} \widetilde{f}_i'(0) = 0$ (see property (iii)).

Moreover, we have

$$\widetilde{f}_i(x) - \widetilde{f}_j(-x) = \begin{cases} f_i(x) - (1-pm) f_j(x) - \sum_{k\in I} p f_k(x), & \text{if } x \ge 0, \\ (1-pm) f_i(-x) + \sum_{k\in I} p f_k(-x) - f_j(-x), & \text{if } x \le 0. \end{cases} \tag{3.8}$$

The oddness of $\widetilde{f}_i(x) - \widetilde{f}_j(-x)$ is attained for $p = 2/m$ (see property (iv)), leading to the extended function:

$$\widetilde{f}_i(x) := \begin{cases} f_i(x), & \text{if } x \ge 0, \\ -f_i(-x) + \sum_{j\in I} \frac{2}{m} f_j(-x), & \text{if } x \le 0. \end{cases} \tag{3.9}$$

The constructed functions $\widetilde{f}_i$ possess the desired properties, completing the proof. ∎

Building upon the even-odd extension Lemma 3.4, we outline Mugnolo and Rhandi's proof [MR22] of Theorem 3.3.

Proof [Proof sketch] Let $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$ be given. We proceed in several steps to construct a classical solution to (P_L) with initial data $\boldsymbol{f}$.

Step 1: Initially, the problem (P_L) is considered on $\boldsymbol{\mathcal{S}_m^n}$, defined as

$$\boldsymbol{\mathcal{S}_m^n} := B_{\mathcal{S}_m}(\mathbf{0}, n), \quad n \in \mathbb{N}, \tag{3.10}$$

where $B_{\mathcal{S}_m}(\mathbf{0}, n)$ denotes the ball centered at the central vertex $\mathbf{0}$ with radius n in the metric space $(\boldsymbol{\mathcal{S}_m}, d_{\mathcal{S}_m})$. In other words, $\boldsymbol{\mathcal{S}_m^n}$ is the subgraph of $\boldsymbol{\mathcal{S}_m}$ consisting of the central vertex $\mathbf{0}$ and m open edges of length n emanating from it.

The authors formulate the problem (P_L) on $\boldsymbol{\mathcal{S}_m^n}$ with initial data $\boldsymbol{f}|_{\mathcal{S}_m^n}$ where the endpoints (n, i), $i \in I$, are subject to Dirichlet BCs.

For each $n \in \mathbb{N}$ and $i \in I$, they study the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u_i^n(t,\cdot) = \widetilde{L} u_i^n(t,\cdot), & t > 0, \\ u_i^n(t,0) = u_i^n(t,n), & t > 0, \\ u_i^n(0,x) = \widetilde{f}_i(x), & x \in (-n,n), \end{cases} \tag{3.11}$$

where $\widetilde{f}_i$ is the extended function defined in (3.4). By classical results for parabolic Cauchy problems on bounded domains (see, e.g., [LB06, Theorem 9.4.1]), we deduce that the aforementioned problem admits a unique solution u_i^n such that

$$\begin{aligned} u_i^n &\in C\big([0,\infty) \times (-n,n)\big), \\ u_i^n &\in C_{\text{loc}}^{1+\frac{\nu}{2}, 2+\nu}\big((0,\infty) \times [-n,n]\big), \end{aligned} \quad i \in I, \nu \in (0,1). \tag{3.12}$$

Step 2: We define a function $\hat{\boldsymbol{u}}^n$ on $[0,\infty) \times \boldsymbol{\mathcal{S}_m^n}$ by

$$\hat{\boldsymbol{u}}^n(t, \boldsymbol{x_i}) := u_i^n(t, |\boldsymbol{x_i}|), \quad i \in I. \tag{3.13}$$

For establishing that $\hat{\boldsymbol{u}}^n$ is a classical solution of (P_L) on $\boldsymbol{\mathcal{S}_m^n}$, it remains to show that $\hat{\boldsymbol{u}}^n(t,\cdot) \in \mathrm{Dom}(\boldsymbol{L}) \cap C_b(\boldsymbol{\mathcal{S}_m^n})$ for all $t > 0$, i.e.,

$$\hat{\boldsymbol{u}}^n(t, \mathbf{0}_k) = \hat{\boldsymbol{u}}^n(t, \mathbf{0}_l) \quad \text{for all } k, l \in I \quad \text{and} \tag{3.14}$$

$$\sum_{k \in I} \partial_{\boldsymbol{x}_k} \hat{\boldsymbol{u}}^n(t, \mathbf{0}_k) = 0, \quad t > 0. \tag{3.15}$$

Using the properties of the extended functions $\widetilde{f}_i$ defined in Lemma (3.4), one can show that the solutions u_i^n of the truncated problems (3.11) satisfy the necessary parity conditions, which in turn imply the continuity (3.14) and Kirchhoff VCs (3.15) for $\hat{\boldsymbol{u}}^n$.

First, recall we have defined the functions $\widetilde{f}_{ij}(x) := \widetilde{f}_i(x) - \widetilde{f}_j(-x)$ for $i, j \in I$, which are odd by construction (see property (iv)). We have then that the function $w_{ij}^n(t,x) := u_i^n(t,x) - u_j^n(t,-x)$ is the unique solution of the truncated problem (3.11) with initial data $\widetilde{f}_{ij}$. Due to the invariance of the operator $\widetilde{L}$ under the transformation $x \mapsto -x$ and the oddness of $\widetilde{f}_{ij}$, we deduce that $w_{ij}^n(t,x)$ is odd in x for all $t \geq 0$. Hence, $w_{ij}^n(t,0) = 0$, $\forall t \geq 0$.

Translating this property back to the original functions u_i^n, we have

$$u_i^n(t,0) - u_j^n(t,0) = w_{ij}^n(t,0) = 0, \quad \forall t \geq 0, \forall i,j \in I, \tag{3.16}$$

which implies $\hat{\boldsymbol{u}}^n(t,\cdot)$ satisfies (3.14).

Next, consider the sum $\sigma^n(t,x) := \sum_{k\in I} u_k^n(t,x)$. By Lemma (3.4), we know that the sum $\sum_{i\in I} \widetilde{f}_i$ is an even function (see property (ii)). Therefore, $\sigma^n(t,x)$, which is the unique solution of the truncated problem (3.11) with initial data $\sum_{i=1}^{m} \widetilde{f}_i(x)$, is even in x for all $t \geq 0$. Consequently, its derivative $\partial_x \sigma^n(t,x)$ is odd in x for all $t \geq 0$ (see property (iii)), and in particular, satisfies

$$\partial_x \sigma^n(t,0) = 0, \quad \forall t \geq 0. \tag{3.17}$$

Expressing this condition in terms of the original functions u_i^n, we obtain

$$\sum_{i\in I} \partial_x u_i^n(t,0) = \partial_x \sigma^n(t,0) = 0, \quad \forall t \geq 0, \tag{3.18}$$

which is precisely the Kirchhoff VC (3.15).

Step 3: Using Schauder estimates and a compactness argument, as done in [MR22] taking the limit for $n \to \infty$ yields a classical solution $\boldsymbol{u}$ of the parabolic problem $(P_{\boldsymbol{L}})$ on the star graph $\boldsymbol{\mathcal{S}_m}$. The Schauder estimates (see [LB06, Theorem 2.2.1]) provide uniform bounds and regularity properties for the solutions u_i^n of the truncated problems (3.11). These estimates ensure that the solutions u_i^n are uniformly bounded and equicontinuous on compact subsets of $(0,\infty) \times (0,\infty)$. By the Arzelà–Ascoli theorem and a diagonal argument, the authors extract a subsequence $(u_i^{n_k})_{k\in\mathbb{N}}$ that converges locally uniformly, along with its derivatives up to order 2, to a function $u_i \in C_{\text{loc}}^{1+\nu/2,2+\nu}((0,\infty) \times [0,\infty))$. The limit function u_i satisfies the differential equation $\partial_t u_i = \widetilde{L} u_i$ on $(0,\infty) \times (0,\infty)$ and the initial condition $u_i(0,\cdot) = \widetilde{f}_i$ on $[0,\infty)$. Moreover, the Standard VCs for u_i^n, while passing to the limit for $n \to \infty$, ensure that the limit function $u_i(t,|\boldsymbol{x}_i|)$ satisfies the Standard VC required for a classical solution of $(P_{\boldsymbol{L}})$ on the star graph $\boldsymbol{\mathcal{S}_m}$. Consequently, the function $\boldsymbol{u}(t,\boldsymbol{x}_i) := u_i(t,|\boldsymbol{x}_i|)$ constitutes a classical solution to $(P_{\boldsymbol{L}})$ on the star graph $\boldsymbol{\mathcal{S}_m}$.

Step 4: We now derive the representation formula for the semigroup $(\boldsymbol{T_m}(t))_{t\geq 0}$ defined by

$$\big(\boldsymbol{T_m}(t)\boldsymbol{f}\big)(\boldsymbol{x}_i) := u_i(t,|\boldsymbol{x}_i|) = \boldsymbol{u}(t,\boldsymbol{x}_i), \quad i \in I,\ t \geq 0, \tag{3.19}$$

where u_i is the limit function and u is the classical solution to $(P_{\boldsymbol{L}})$ obtained in step 3.

To derive the representation formula (3.1) for the semigroup $\big(\boldsymbol{T_m}(t)\big)_{t\geq 0}$, [MR22] make use of the general existence and representation theorem for parabolic problems on $\mathbb{R}$ [LB06, Theorem 2.2.5]. They consider the parabolic problem $(P_{\widetilde{L}})$ on $\mathbb{R}$ with initial condition $\widetilde{f}_i$, where $\widetilde{f}_i$ are the extended functions defined in (3.3).

By [LB06, Theorem 2.2.5], there exists a semigroup of linear operators $(T(t))_{t\geq 0}$ acting on $C_b(\mathbb{R})$ with the property that the solution of $(P_{\widetilde{L}})$ is given by

$$u_i(t,x) = (T(t)\widetilde{f}_i)(x), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{3.20}$$

with an integral representation:

$$\big(T(t)\widetilde{f}_i\big)(x) = \int_{\mathbb{R}} \kappa(t,x,y)\widetilde{f}_i(y)\mathrm{d}y, \tag{3.21}$$

for some kernel κ. The kernel κ satisfies the following properties:

- κ is strictly positive;
- $\kappa(t,\cdot,\cdot)$ and $\kappa(t,x,\cdot)$ are measurable for any $t > 0$ and $x \in \mathbb{R}$;
- for almost every $y \in \mathbb{R}$, the function $\kappa(\cdot,\cdot,y)$ belongs to the space $C^{1+\nu/2,2+\nu}_{\mathrm{loc}}((0,\infty)\times\mathbb{R})$ and is a solution of the equation $\partial_t u - \widetilde{L}u = 0$.

From definition 3.19 and equations (3.20) and (3.21), we have:

$$\big(\boldsymbol{T_m}(t)\boldsymbol{f}\big)(\boldsymbol{x}_i) = (T(t)\widetilde{f}_i)(|\boldsymbol{x}_i|) = \int_{\mathbb{R}} \kappa(t,|\boldsymbol{x}_i|,y)\widetilde{f}_i(y)\mathrm{d}y.$$

Applying the definition of $\widetilde{f}_i$ (3.3), we obtain:

$$\begin{aligned}(\boldsymbol{T_m}(t)\boldsymbol{f})(\boldsymbol{x}_i) &= \int_{\mathbb{R}+} \kappa(t,|\boldsymbol{x}_i|,y)\widetilde{f}_i(y)\mathrm{d}y + \int_{\mathbb{R}-} \kappa(t,|\boldsymbol{x}_i|,y)\widetilde{f}_i(y)\mathrm{d}y \\ &= \int_{\boldsymbol{r}_i} \big(\kappa(t,|\boldsymbol{x}_i|,|\boldsymbol{y}_i|) - \kappa(t,|\boldsymbol{x}_i|,-|\boldsymbol{y}_i|)\big)\boldsymbol{f}(\boldsymbol{y}_i)\,\mathrm{d}\boldsymbol{y}_i \\ &\quad + \frac{2}{m}\sum_{j\in I}\int_{\boldsymbol{r}_j} \kappa(t,|\boldsymbol{x}_i|,-|\boldsymbol{y}_j|)\boldsymbol{f}(\boldsymbol{y}_j)\,\mathrm{d}\boldsymbol{y}_j.\end{aligned}$$

This representation formula characterizes the semigroup $(\boldsymbol{T_m}(t))_{t\geq 0}$ generated by $\boldsymbol{L}$ on $C_b(\boldsymbol{\mathcal{S}_m})$. It encodes the interplay between the star graph structure and the dynamics induced by $\boldsymbol{L}$, fully describing the evolution of solutions to $(P_{\boldsymbol{L}})$.

The conservation property of the semigroup $\big(\boldsymbol{T_m}(t)\big)_{t\geq 0}$ is established under the assumptions that the coefficient function c in the operator $\boldsymbol{L}$ vanishes identically, and that $\big(T(t)\big)_{t\geq 0}$ satisfies $T(t_0)\mathbb{1} = \mathbb{1}$ for some $t_0 > 0$. The proof relies on the representation formula (3.5) for $\boldsymbol{T_m}(t)\boldsymbol{f}$, which involves the integral kernel κ of $\big(T(t)\big)_{t\geq 0}$.

According to [LB06, Proposition 4.1.10], the hypothesis $T(t_0)\mathbb{1} = \mathbb{1}$ for some $t_0 > 0$ implies that $T(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$. Consequently, the integral kernel κ satisfies

$$\int_{\mathbb{R}} \kappa(t,x,y)\,\mathrm{d}y = 1, \quad \forall t > 0, \quad \forall x \in \mathbb{R}, \tag{3.22}$$

which implies

$$\int_{\boldsymbol{r}_i} \kappa(t,|\boldsymbol{x}_i|,|\boldsymbol{y}_i|)\,\mathrm{d}\boldsymbol{y}_i + \int_{\boldsymbol{r}_j} \kappa(t,|\boldsymbol{x}_i|,-|\boldsymbol{y}_j|)\,\mathrm{d}\boldsymbol{y}_j = 1, \qquad \forall i,j \in I, \quad \forall \boldsymbol{x}_i \in \boldsymbol{\mathcal{S}_m}, \quad \forall t > 0. \tag{3.23}$$

Substituting (3.23) into (3.1) reveals that $\boldsymbol{T_m}(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$, establishing the conservation property of the semigroup $\big(\boldsymbol{T_m}(t)\big)_{t\geq 0}$. A proof of this conservation property is given in Lemma A.1 in the appendix.

In conclusion, the sketch of the proof of (3.3) in [MR22] is now complete. We have established the existence of a classical solution to the parabolic problem (P_L) on the star graph $\boldsymbol{\mathcal{S}_m}$, which is unique if the solution to $(P_{\widetilde{L}})$ is unique. We have derived the explicit representation formula for the associated semigroup $\big(\boldsymbol{T_m}(t)\big)_{t\geq 0}$. Furthermore, we have proven the conservation property of the semigroup $\big(\boldsymbol{T_m}(t)\big)_{t\geq 0}$.[8] ∎

The strategy employed in the proof of Theorem 3.3 provides a general framework for studying parabolic problems on metric graphs by reducing the analysis to the study of an associated problem on the real line. The main components of this approach are: first, the construction of even-odd extensions of the initial data, as described in Lemma 3.4, which ensures that the extended problem inherits the necessary symmetry properties; and second, the utilization of the invariance of the extended operator $\widetilde{L}$ under the transformation $x \mapsto -x$, which implies that the solutions to the extended problem possess the same parity properties as the initial data.

The methodology presented in the proof of Theorem 3.3 is not restricted to the specific case of Standard VCs at the vertices of the metric graph. Given the existence of an integral representation for the associated semigroup, the technique might be generalized to encompass other types of vertex conditions and more general graph topologies. This adaptability highlights the potential for applying this approach to a broad spectrum of parabolic problems on metric graphs, offering new opportunities for further investigation in this field.

3.3 EXPLICIT REPRESENTATION AND PROPERTIES OF THE ORNSTEIN-UHLENBECK SEMIGROUP ON $\boldsymbol{\mathcal{S}_m}$

To better understand the application of Theorem 3.3 to the operator $\mathcal{A}$ (2.21) on metric star graphs, we first introduce the kernel on $\mathbb{R}$ of the associated semigroup and which is given by (see (2.5)):

$$\kappa_{OU}(t,x,y) := \gamma_{\tau x, \frac{1-\tau^2}{2}}(y) = \frac{1}{\sqrt{\pi(1-\tau^2)}} \exp\left[-\frac{(\tau x - y)^2}{1-\tau^2}\right], \tag{3.24}$$

where $\tau := e^{-t}$, for $t > 0$ and $x, y \in \mathbb{R}$.

This kernel is strictly positive and satisfies the semigroup property. Moreover, it is the fundamental solution of the equation

$$\partial_t \kappa_{OU}(t,x,y) = \frac{1}{2}\partial_{xx}\kappa_{OU}(t,x,y) - x\partial_x \kappa_{OU}(t,x,y), \quad t > 0, \quad x, y \in \mathbb{R}.$$

8 The conservation property has important consequences, such as the conservation of total *mass* or *probability* in diffusion processes on metric graphs, ensuring that the total amount of *the diffusing substance* remains constant over time.

We have then (2.5)

$$(OU(t)f)(x) = \int_{\mathbb{R}} \kappa_{OU}(t,x,y)\phi(y)\,\mathrm{d}y, \quad t \geq 0, \quad x \in \mathbb{R}, \quad f \in C_b(\mathbb{R}). \tag{3.25}$$

The OU operator $\mathcal{A}$ on $\boldsymbol{\mathcal{S}_m}$, with $q(x) = \frac{1}{2}$, $b(x) = -x$, and $c(x) = 0$, satisfies the hypotheses of Theorem 3.3. Consequently, we obtain the following corollary, which establishes the existence and uniqueness of classical solutions to the associated parabolic problem on $\boldsymbol{\mathcal{S}_m}$ and provides a representation formula for the semigroup in terms of the OU kernel on $\mathbb{R}$.

Corollary 3.5 [MR22, Corollary 2.4] *For every $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$, there exists a unique bounded classical solution u of the parabolic problem*

$$\begin{cases} \partial_t \boldsymbol{u}(t,\cdot) = \boldsymbol{A}\boldsymbol{u}(t,\cdot), & t > 0, \\ \boldsymbol{u}(0,\cdot) = \boldsymbol{f}(\cdot), \end{cases} \tag{3.26}$$

given by the action of the semigroup $\big(\boldsymbol{OU_m}(t)\big)_{t\geq 0}$ on $\boldsymbol{\mathcal{S}_m}$. The semigroup acts as

$$\begin{aligned}(\boldsymbol{OU_m}(t)\boldsymbol{f})(\boldsymbol{x}_i) := \frac{1}{\sqrt{\pi(1-\tau^2)}} \int_0^\infty \big(\kappa_{OU}(t,|\boldsymbol{x}_i|,|\boldsymbol{y}_i|) - \kappa_{OU}(t,|\boldsymbol{x}_i|,-|\boldsymbol{y}_i|)\big)\boldsymbol{f}(\boldsymbol{y}_i)\,\mathrm{d}\boldsymbol{y}_i \\ + \frac{2}{m\sqrt{\pi(1-\tau^2)}} \sum_{j\in I} \int_0^\infty \kappa_{OU}(t,|\boldsymbol{x}_i|,-|\boldsymbol{y}_j|)\boldsymbol{f}(\boldsymbol{y}_j)\,\mathrm{d}\boldsymbol{y}_j\end{aligned} \tag{3.27}$$

where $\tau := e^{-t}$, for $i \in I$. Furthermore, $\big(\boldsymbol{OU_m}(t)\big)_{t\geq 0}$ possesses the following properties:

- ***Conservative**: As defined in Theorem 3.3, $\boldsymbol{OU_m}(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$.*
- ***Contractive**: $\|\boldsymbol{OU_m}(t)\boldsymbol{f}\|_\infty \leq \|\boldsymbol{f}\|_\infty$ for all $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$ and $t \geq 0$. This property implies that the semigroup does not increase the supremum norm of the initial condition.*
- ***Irreducible**: For any non-negative, non-zero function $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$ and any $\boldsymbol{x} \in \boldsymbol{\mathcal{S}_m}$, there exists $t > 0$ such that $\boldsymbol{OU_m}(t)\boldsymbol{f}(x) > 0$. This property ensures that the semigroup spreads the* mass *or* probability *of the initial condition to all points of the metric star graph.*
- ***Strong Feller property**: For any bounded Borel measurable function $\boldsymbol{f} : \boldsymbol{\mathcal{S}_m} \to \mathbb{R}$, the function $\boldsymbol{OU_m}(t)\boldsymbol{f}$ is continuous on $\boldsymbol{\mathcal{S}_m}$ for all $t > 0$. This property implies that the semigroup improves the regularity of the initial condition.*

The representation formula (3.27) for the semigroup on $\boldsymbol{\mathcal{S}_m}$ and the properties of the semigroup follow directly from the general formula in Theorem 3.3, with the integral kernel κ replaced by the kernel κ_{OU} given in (3.24).

Remark 3.6 *The kernel κ_{OU} satisfies the inequality*

$$\kappa_{OU}(t,|\boldsymbol{x}_i|,|\boldsymbol{y}_i|) \geq \kappa_{OU}(t,|\boldsymbol{x}_i|,-|\boldsymbol{y}_i|), \quad t > 0, \quad \boldsymbol{x}_i, \boldsymbol{y}_i \in \boldsymbol{\mathcal{S}_m}. \tag{3.28}$$

The inequality, together with the representation formula (3.27), implies that the semigroup $\big(\boldsymbol{OU_m}(t)\big)_{t\geq 0}$ on $\boldsymbol{\mathcal{S}_m}$ is positive, i.e., for any non-negative function $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$, we have $\boldsymbol{OU_m}(t)\boldsymbol{f} \geq 0$ for all $t \geq 0$.

3.3.1 *Properties of* $\boldsymbol{OU_m}(t)$ *on* $\boldsymbol{\mathcal{S}_m}$

In this subsection, we present a comprehensive list of properties of the semigroup $\left(\boldsymbol{OU_m}(t)\right)_{t\geq 0}$ on the metric star graph $\boldsymbol{\mathcal{S}_m}$. These properties, derived from the representation formula (3.27) and the connection to the classical semigroup on $\mathbb{R}$, provide a characterization of the semigroup's action on various function spaces and its long-time behaviour.

Proposition 3.7 *The semigroup* $\left(\boldsymbol{OU_m}(t)\right)_{t\geq 0}$ *on* $\boldsymbol{\mathcal{S}_m}$ *satisfies the following properties:*

1. *Positivity, as noted in Remark 3.6.*
2. *Contractivity, conservativity and irreducibility on* $C_b(\boldsymbol{\mathcal{S}_m})$*, and strong Feller property, as established in Corollary 3.5 .*
3. *Extrapolation to a consistent family of* C_0*-semigroups*[9] *on* $L^p(\boldsymbol{\mathcal{S}_m})$ *where* $p \in [1,\infty)$*. Consequently, the semigroup* $\left(\boldsymbol{OU_m}(t)\right)_{t\geq 0}$ *admits a unique extension to a strongly continuous semigroup on* $L^p(\boldsymbol{\mathcal{S}_m})$ *for all* $p \in [1,\infty)$*, and these extensions are consistent in the sense that they coincide on the intersections of the* $L^p(\boldsymbol{\mathcal{S}_m})$ *spaces.*
4. *Invariance of* $C_0(\boldsymbol{\mathcal{S}_m})$*, the space of continuous functions vanishing at infinity. In other words, if* $f \in C_0(\boldsymbol{\mathcal{S}_m})$*, then* $\boldsymbol{OU_m}(t)f \in C_0(\boldsymbol{\mathcal{S}_m})$ *for all* $t \geq 0$*.*
5. *Lack of strong continuity on* $C_b(\boldsymbol{\mathcal{S}_m})$*. More specifically,* $\lim\limits_{t\to 0} \|\boldsymbol{OU_m}(t)f - f\|_\infty = 0$ *if and only if* $f \in BUC(\boldsymbol{\mathcal{S}_m})$*, and* $\lim\limits_{t\to 0} |f(e^{-t}x_i) - f(x_i)| = 0$ *uniformly with respect to* $x_i \in \boldsymbol{\mathcal{S}_m}$*, where* $BUC(\boldsymbol{\mathcal{S}_m})$ *refers to the space of all functions on* $\boldsymbol{\mathcal{S}_m}$ *that are both bounded and uniformly continuous.*

The extrapolation, invariance of $C_0(\boldsymbol{\mathcal{S}_m})$ and lack of strong continuity on $C_b(\boldsymbol{\mathcal{S}_m})$ as stated in [MR22, Remark 4.3], can be established using the same arguments as in the case of the classical Ornstein-Uhlenbeck semigroup on $\mathbb{R}$, which are presented in [LB06, Sections 9.2 and 9.4]. This is possible because the Ornstein–Uhlenbeck semigroup on the metric star graph $\boldsymbol{\mathcal{S}_m}$ is constructed using the Ornstein–Uhlenbeck kernel on $\mathbb{R}$, as seen in the representation formula (3.27). The properties of the kernel, such as its strict positivity and the semigroup property, are inherited by the semigroup on $\boldsymbol{\mathcal{S}_m}$.

Next, we discuss an interesting connection between the semigroup $\left(\boldsymbol{OU_m}(t)\right)_{t\geq 0}$ and scattering theory, based on [MR22, Remark 4.2].

Remark 3.8 [MR22, Remark 4.2] *The representation formula* (3.27) *for the semigroup* $\left(\boldsymbol{OU_m}(t)\right)_{t\geq 0}$ *on* $\boldsymbol{\mathcal{S}_m}$ *can be rewritten as follows:*

$$\begin{aligned}(\boldsymbol{OU_m}(t)\boldsymbol{f})(\boldsymbol{x}_i) &= \frac{1}{\sqrt{\pi(1-\tau^2)}}\int_{\boldsymbol{r}_i} e^{-\frac{(\tau|\boldsymbol{x}_i|-|\boldsymbol{y}_i|)^2}{1-\tau^2}} f(\boldsymbol{y}_i)\,\mathrm{d}\boldsymbol{y}_i \\ &+ \frac{1}{\sqrt{\pi(1-\tau^2)}}\sum_{j\in I}\int_{\boldsymbol{r}_j}\sigma_{ij}e^{-\frac{(\tau|\boldsymbol{x}_i|+|\boldsymbol{y}_j|)^2}{1-\tau^2}} f(\boldsymbol{y}_j)\,\mathrm{d}\boldsymbol{y}_j, \end{aligned} \tag{3.29}$$

9 Strongly continuous semigroups.

where $\tau := e^{-t}$ and $\Sigma = (\sigma_{ij})$ is the scattering matrix defined by

$$\sigma_{ij} := \begin{cases} \frac{2-m}{m}, & \text{if } i = j, \\ \frac{2}{m}, & \text{otherwise.} \end{cases} \tag{3.30}$$

The representation (3.29) has a natural interpretation in terms of scattering theory. The first term represents the probability amplitude for a particle to propagate freely between two points on the same ray without undergoing scattering. The second term accounts for scattering processes, encompassing both transitions between distinct rays (with a weight of $\frac{2}{m}$) and reflections at the central vertex (with a weight of $\frac{2-m}{m}$). That is, the scattering matrix Σ encodes the transmission and reflection coefficients at $\mathbf{0}$.

3.4 INVARIANT MEASURE AND SPECTRAL PROPERTIES OF THE GENERATOR

We now focus on the invariant measure of $\big(\boldsymbol{OU_m}(t)\big)_{t\geq 0}$ and the spectral characteristics of its generator on the metric star graph $\boldsymbol{\mathcal{S}_m}$.

3.4.1 *Existence and Uniqueness of the Invariant Measure*

As defined in Section 2.1.1, an invariant measure is a probability measure invariant under the action of the semigroup. The invariant measure is essential for understanding the long-time behavior of the semigroup and the associated parabolic problem.

Theorem 3.9 [MR22, Theorem 4.4] *The OU semigroup $\big(\boldsymbol{OU_m}(t)\big)_{t\geq 0}$ admits a unique invariant probability measure, denoted by $\boldsymbol{\gamma_m}$. This measure has a density*

$$\boldsymbol{\gamma_m}(\mathrm{d}\boldsymbol{x}_i) = \frac{2}{m\sqrt{\pi}} e^{-|\boldsymbol{x}_i|^2}\,\mathrm{d}\boldsymbol{x}_i, \quad i \in I, \tag{3.31}$$

under the Lebesgue measure on $\boldsymbol{\mathcal{S}_m}$.

Proof Let $\boldsymbol{f} \in C_b(\boldsymbol{\mathcal{S}_m})$, and let $\tilde{f}_i \in C_b(\mathbb{R})$, $i \in I$.[10] Furtermore, let $OU(\cdot)$ denote the OU semigroup on $\mathbb{R}$, with γ representing the Gaussian measure on $\mathbb{R}$, i. e., $\gamma(\mathrm{d}x) = \frac{1}{\sqrt{\pi}} e^{-|x|^2}\mathrm{d}x$. Finally, let $\boldsymbol{OU_m}(\cdot)$ be the OU semigroup on $\boldsymbol{\mathcal{S}_m}$.

Given that γ is the invariant measure of $OU(\cdot)$ on $\mathbb{R}$ and that both $\sum_{i\in I} \tilde{f}_i$ and and, consequently $\left(OU(t) \sum_{i\in I} \tilde{f}_i\right)$ are even functions; we have

$$\int_{\mathbb{R}^+} \left(OU(t) \sum_{i\in I} \tilde{f}_i\right)(x)\,\gamma(\mathrm{d}x) = \int_{\mathbb{R}^+} \sum_{i\in I} \tilde{f}_i(x)\,\gamma(\mathrm{d}x). \tag{3.32}$$

10 We are following the notation of Lemma 3.4

Using the definition of $\boldsymbol{OU_m}(\cdot)$ and the invariance property (3.32), we obtain

$$\begin{aligned}\int_{\boldsymbol{\mathcal{S}_m}} \boldsymbol{OU_m}(t)\boldsymbol{f}(\boldsymbol{x})\,\boldsymbol{\gamma}(\mathrm{d}\boldsymbol{x}) &= \sum_{i\in I}\int_{\boldsymbol{r}_i} \boldsymbol{OU_m}(t)\boldsymbol{f}(\boldsymbol{x}_i)\,\boldsymbol{\gamma}(\mathrm{d}\boldsymbol{x}_i)\\ &= \sum_{i\in I}\int_{\mathbb{R}_+}\left(OU(t)\tilde{f}_i\right)(x)\,\gamma(\mathrm{d}x) = \int_{\mathbb{R}_+}\left(OU(t)\sum_{i\in I}\tilde{f}_i\right)(x)\,\gamma(\mathrm{d}x)\\ &= \int_{\mathbb{R}_+}\sum_{i\in I}\tilde{f}_i(x)\,\gamma(\mathrm{d}x) = \sum_{i\in I}\int_{\boldsymbol{r}_i}\boldsymbol{f}(\boldsymbol{x}_i)\,\boldsymbol{\gamma}(\mathrm{d}\boldsymbol{x}_i) = \int_{\boldsymbol{\mathcal{S}_m}}\boldsymbol{f}(\boldsymbol{x})\,\boldsymbol{\gamma}(\mathrm{d}\boldsymbol{x}).\end{aligned}$$

This shows that $\boldsymbol{\gamma}$ is an invariant measure for $\boldsymbol{OU_m}(\cdot)$. After normalization, we obtain that $\boldsymbol{\gamma_m}$ is both invariant and a probability measure for $\boldsymbol{OU_m}(\cdot)$.

The uniqueness of $\boldsymbol{\gamma_m}$ follows directly from the ergodicity of the invariant measure (see [LB06, Theorem 8.1.15]). ∎

The unique invariant measure $\boldsymbol{\gamma_m}$ have important consequences for the long-time behaviour of $\left(\boldsymbol{OU_m}(t)\right)_{t\geq 0}$ and the solutions to the associated parabolic problem (3.26). In particular, it implies the ergodicity of the semigroup, meaning that the solutions converge to the equilibrium state described by the invariant measure as time tends to infinity.

3.4.2 *Spectral Properties of the Realization $\boldsymbol{A}_2$ in $L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m})$*

We now focus on the spectral properties of the realization $\boldsymbol{A}_2$ of $\mathcal{A}$ in the Gaussian Lebesgue space $L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m})$, where $\boldsymbol{\gamma}_m$ is the invariant measure defined in (3.31).

To characterize the spectrum of $\boldsymbol{A}_2$, we first require the following lemma, which describes the spectrum of the one-dimensional OU operator on $L^2_{\gamma_1}(\mathbb{R}_+)$ under Dirichlet and Neumann BCs at 0.

Lemma 3.10 [MR22, Lemma 4.7] *The spectrum of the OU operator on $L^2_{\gamma_1}(\mathbb{R}_+)$ is purely discret and comprises the following eigenvalues:*

- $\{-2n : n \in \mathbb{N}_0\}$, *subject to Neumann BCs at* 0,
- $\{-2n-1 : n \in \mathbb{N}_0\}$, *subject to Dirichlet BCs at* 0.

Proof As discussed in Lemma 2.7, the realization A of the OU operator $\mathcal{A}$ on $L^2_\gamma(\mathbb{R})$ has a discrete spectrum consisting of simple eigenvalues $n = 0, -1, -2, \ldots$, with corresponding eigenfunctions H_n.[11]

Since A leaves invariant the subspaces of odd and even $L^2_\gamma(\mathbb{R})$-functions, its spectrum can be decomposed into the spectra of its restrictions to these subspaces. Furthermore, these restrictions are related by unitary equivalence and are isospectral to the Dirichlet and Neumann realizations A_D and A_N of $\mathcal{A}$ on $L^2_{\gamma_1}(\mathbb{R}_+)$.

We have $H_n'(0) = 0$ if and only if n is even, while $H_n(0) = 0$ if and only if n is odd. Therefore, H_n serves as an eigenfunction of A_N for even n, and as an eigenfunction of A_D for odd n, establishing the claim. ∎

The following theorem, central to our analysis, establishes the spectral properties of $\boldsymbol{A}_2$.

11 Here, H_n denotes the n-th Hermite polynomial.

Theorem 3.11 *The spectrum of the operator $\boldsymbol{A}_2$ is purely discrete and consists of the eigenvalues*

$$\sigma(\boldsymbol{A}_2) = \{-n : n \in \mathbb{N}_0\}.$$

Moreover, each even eigenvalue has multiplicity 1, while each odd eigenvalue has multiplicity $m-1$.

Proof Following [MR22], we adapt the method outlined in [Mal13, Section 3.5] for the OU operator on the metric star graph $\boldsymbol{\mathcal{S}_m}$.

First, observe that $\boldsymbol{A}_2$ leaves invariant the mutually orthogonal subspaces L^2_{odd} and L^2_{even} of odd and even $L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m})$-functions, respectively. This can be proved following the ideas in [Mal13, Section 3.5], with minor modifications to adapt the proof to the OU operator.

Moreover, $\boldsymbol{A}_2$ commutes with the bounded, unitary operator $\boldsymbol{R}$ on $L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m})$ defined by

$$\boldsymbol{R} : (f_1, \ldots, f_{m-1}, f_m) \mapsto (f_2, \ldots, f_m, f_1).$$

To see this, let $f = (f_1, \ldots, f_m) \in D(\boldsymbol{A}_2)$. Then, $\boldsymbol{R}f = (f_2, \ldots, f_m, f_1) \in D(\boldsymbol{A}_2)$ because the Kirchhoff condition $\sum_{i\in I}(\boldsymbol{R}f)'_i(0) = \sum_{i\in I} f'_i(0) = 0$ is satisfied, and $\boldsymbol{A}_2(\boldsymbol{R}f) = \boldsymbol{R}(\boldsymbol{A}_2 f)$.

By the Spectral Theorem for normal operators, the fact that $\boldsymbol{A}_2$ and $\boldsymbol{R}$ commute implies that they can be simultaneously diagonalized. Equivalently, the eigenfunctions of $\boldsymbol{A}_2$ and $\boldsymbol{R}$ coincide.

The eigenspaces associated with $\boldsymbol{R}$ are

$$E_j := (1, z^j, z^{2j}, \ldots, z^{j(m-1)}) \otimes L^2_{\boldsymbol{\gamma_m}}(\mathbb{R}^+), \quad j = 0, \ldots, m-1,$$

where $z := e^{2\pi i/m}$. Indeed, for any $f \in L^2_{\boldsymbol{\gamma_m}}(\mathbb{R}^+)$ and $j = 0, \ldots, m-1$, we have

$$\boldsymbol{R}((1, z^j, z^{2j}, \ldots, z^{j(m-1)}) \otimes f) = z^j((1, z^j, z^{2j}, \ldots, z^{j(m-1)}) \otimes f),$$

which shows that $(1, z^j, z^{2j}, \ldots, z^{j(m-1)}) \otimes f$ is an eigenfunction of $\boldsymbol{R}$ with eigenvalue z^j.

Observe that $E_0 = L^2_{\text{even}}$ and $L^2_{\text{odd}} = \bigoplus_{j=1}^{m-1} E_j$. This follows from the fact that the eigenfunctions in E_0 are even, while those in E_j for $j \in \{1, \ldots, m-1\}$ are odd. Restricting $\boldsymbol{A}_2$ to L^2_{odd} enforces Dirichlet BCs at 0. By Lemma 3.10, the spectrum of this restriction is $\{-2n-1 : n \in \mathbb{N}_0\}$, each eigenvalue having multiplicity $m-1$. Similarly, restricting $\boldsymbol{A}_2$ to $E_0 = L^2_{\text{even}}$ corresponds to Neumann BC at 0. Again, by Lemma 3.10, the spectrum of this restriction is $\{-2n : n \in \mathbb{N}_0\}$, each eigenvalue having multiplicity 1.

Combining these results yields the claimed spectrum of $\boldsymbol{A}_2$. ∎

3.5 POTENTIAL EXTENSIONS AND GENERALIZATIONS

The results presented in this section, based primarily on the work of Mugnolo and Rhandi [MR22], provide a comprehensive analysis of the OU semigroup $\big(\boldsymbol{OU_m}(t)\big)_{t\geq 0}$ and its generator $\boldsymbol{A_2}$ on metric star graphs $\boldsymbol{\mathcal{S}_m}$. The existence and uniqueness of the invariant measure $\boldsymbol{\gamma_m}$, along with the spectral characteristics of the generator $\boldsymbol{A_2}$

and the semigroup $\big(\boldsymbol{OU_m}(t)\big)t \geq 0$, shed light on the interplay between the graph structure and the long-time behaviour of the associated parabolic problem (3.26).

The findings in [MR22] focus on metric star graphs with Standard VCs at the central vertex—a particular case of the more general class characterized by Theorem 2.20. A natural question arises: to what extent can the results of [MR22] be extended to metric star graphs with, for example, arbitrary δ-Coupling VCs?

The next chapter extends our analysis to the OU operator $\mathcal{A}$ with δ-Coupling VCs on $\boldsymbol{\mathcal{S}_m}$. We establish a connection between the operator and a singular Sturm-Liouville problem, yielding explicit characterizations of the spectrum and eigenfunctions for Standard and δ-Coupling cases.

4

SPECTRAL PROPERTIES OF THE ORNSTEIN-UHLENBECK OPERATOR WITH δ-COUPLING VERTEX CONDITIONS

... das Wesen der Mathematik liegt gerade in ihrer Freiheit. [1]

— Georg Cantor [Can32]

In this chapter, we explore extending the results of Mugnolo and Rhandi [MR22], focusing on the spectral properties of the Ornstein-Uhlenbeck (OU) operator with δ-Coupling Vertex Conditions (VCs).

We examine the connection between the OU operator and a singular Sturm-Liouville (SL) problem. This perspective allows us to extend the results of [MR22] on eigenfunctions and eigenvalues to include δ-Coupling VCs. Under these generalized conditions, we analyze the spectrum and eigenfunctions for the OU operator $\mathcal{A}$ on $\boldsymbol{\mathcal{S}_m}$.

4.1 STURM-LIOUVILLE FORMULATION OF THE ORNSTEIN-UHLENBECK OPERATOR

In this section, we reformulate the OU operator as a singular SL problem. This approach allows us to apply the well-established theory of SL operators to our spectral analysis. We rely primarily on the treatments of singular SL problems presented in [HK92] and [Wei03].

Recall the OU differential operator $\mathcal{A}$ defined in (2.21). We can transform $\mathcal{A}$ into a SL operator τ via the following identity, valid for sufficiently smooth functions f in the domain of τ (to be precisely defined below):

$$-\mathcal{A}f = -\frac{1}{2}\frac{d^2 f}{dx^2} + x\frac{df}{dx} = \frac{-\left(\frac{1}{2}e^{-x^2}f'\right)'}{e^{-x^2}} =: \tau f. \tag{4.1}$$

The operator τ takes the canonical form of a SL differential expression

$$\frac{-(pf')' + qf}{\gamma_1}$$

[Wei03, Chapter 13], where the coefficients are given by $p(x) := \frac{1}{2}e^{-x^2}$, $q(x) := 0$, and the *weight function* is $\gamma_1(x) := e^{-x^2}$.

We define the maximal operator T generated by τ on $L^2_{\gamma_1}(0,\infty)$ as follows:

$$\begin{aligned} \mathrm{Dom}(T) = \{f \in L^2_{\gamma_1}(0,\infty) : f \text{ and } pf' \text{ are absolutely continuous on } (0,\infty), \\ \tau f \in L^2_{\gamma_1}(0,\infty)\}, \\ Tf = \tau f \text{ for } f \in \mathrm{Dom}(T). \end{aligned} \tag{4.2}$$

1 ... the essence of mathematics lies entirely in its freedom. [Ewa96]

Unless otherwise stated, we assume $f \in \mathrm{Dom}(T)$ for all functions f under consideration for the remainder of this chapter.

Having established the maximal domain for our SL operator, we now turn to its classification. This classification is important for characterizing the self-adjoint restrictions of T, which we will refer to as *self-adjoint realizations* of τ.

Definition 4.1 (Regular and Singular Endpoints) *[Wei03, Chapter 13] An endpoint $x \in \{a, b\}$ is said to be* regular *if it is finite and the functions p^{-1}, q, and w are integrable on some neighbourhood of x within the interval $[a, b]$. Otherwise, x is called a* singular *point.*

In our case, considering the interval $[0, \infty)$, we can deduce that:

- The endpoint $x = 0$ is regular, as it is finite and $p^{-1}(x) = 2e^{x^2}$, $q(x) = 0$, and $w(x) = e^{-x^2}$ are all integrable in a neighborhood of 0.
- The endpoint $x = \infty$ is singular, as it is not finite.

This classification, with one regular endpoint at 0 and one singular endpoint at ∞, establishes our problem within the framework of singular SL theory. Consequently, we can apply results from this theory, particularly [Wei03, Corollary 13.3]. This corollary guarantees the existence and uniqueness of solutions to the initial value problem:

$$(\tau - \lambda)f = g, \quad f(x_0) = y_0, \quad pf'(x_0) = y_1 \tag{4.3}$$

for $g \in L^1_{\mathrm{loc}}(0, \infty)$, $\lambda \in \mathbb{C}$, $x_0 \in (0, \infty)$, and $(y_0, y_1) \in \mathbb{C}^2$. Moreover, the solution $f_\lambda(x)$ is an entire function of λ for each $x \in (0, \infty)$.

We observe that every point in $(-\infty, 0)$ is also a regular point for our differential equation. Consequently, the corollary's validity naturally extends to include $x_0 = 0$. This extension allows us to consider initial conditions at $x = 0$ without additional assumptions or limiting processes.

We now turn our attention to the specific solutions of the homogeneous equation $\tau f = \lambda f$. These solutions are essential for characterizing the spectrum and eigenfunctions of our OU operator on metric star graphs.

4.2 SOLUTIONS TO $\tau f = \lambda f$ FOR $x \in [0, \infty)$

To determine the eigenvalues and eigenfunctions of τ, we first investigate the solutions to the eigenvalue equation $\tau f = \lambda f$:

$$\tau f = -\frac{1}{2}\frac{d^2 f}{dx^2} + x\frac{df}{dx} = \lambda f, \quad x \in [0, \infty). \tag{4.4}$$

Lemma 4.2 *For all $\lambda \in \mathbb{C}$, a fundamental pair of solutions to the equation (4.4) is given by*

$$M\left(-\frac{\lambda}{2}, \frac{1}{2}, x^2\right) \quad \text{and} \quad xM\left(\frac{1-\lambda}{2}, \frac{3}{2}, x^2\right), \tag{4.5}$$

where M is the Kummer function.

Proof The change of variable $z : [0, \infty) \to [0, \infty)$, $z : x \mapsto x^2$ transforms (4.4) into

$$z\frac{d^2 f}{dz^2} + \left(\frac{1}{2} - z\right)\frac{df}{dz} = -\frac{\lambda}{2}f(z). \tag{4.6}$$

This is a special case of Kummer's equation

$$z\frac{d^2f}{dz^2} + (b-z)\frac{df}{dz} - af(z) = 0, \tag{4.7}$$

with $a := -\frac{\lambda}{2}$ and $b := \frac{1}{2}$.[2] The stated fundamental solutions follow from the theory of confluent hypergeometric functions [Mat+22, Table 1], noting that since $b = \frac{1}{2} \notin \mathbb{Z}$, this unique pair covers all possible values of $a = -\frac{\lambda}{2}$, and thus all $\lambda \in \mathbb{C}$. ∎

See Figure (4.1) showing the fundamental pair (4.5) for $\lambda = 4$.

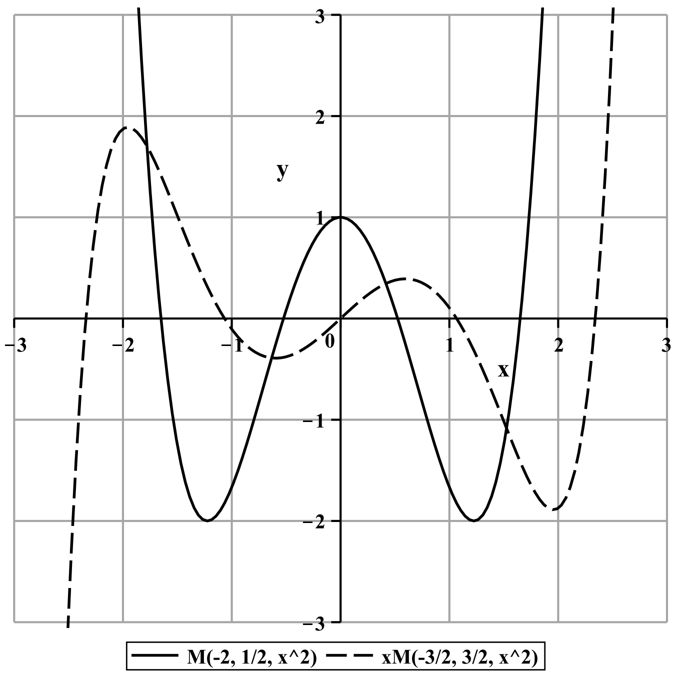

Figure 4.1: Fundamental pair (4.5) for $\lambda = 4$.

4.3 ASYMPTOTIC BEHAVIOR OF KUMMER FUNCTIONS

Understanding the asymptotic behavior of Kummer functions is relevant for determining the eigenvalues of various configurations. From [Mat+22] we have:

$$M(a,b,z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}\left[1 + \mathcal{O}(|z|^{-1})\right], \tag{4.8}$$

$$\text{as } z \to \infty, \text{ for } |\arg(z)| < \frac{\pi}{2}, \text{ for } a \notin \mathbb{Z}^{\leq 0},$$

$$M(-n,b,z) \sim z^n, \quad \text{as } z \to \infty, \text{ for } n \in \mathbb{Z}^{\geq 0}. \tag{4.9}$$

2 The Kummer function $M(a,b,z)$ is defined as a solution to equation (4.7) for complex parameters a, b (with b not a non-positive integer), and complex argument z.

Note that

$$M\left(-n, \frac{1}{2}, z^2\right) = (-1)^n \frac{n!}{(2n)!} H_{2n}(z), \tag{4.10}$$

$$zM\left(-n, \frac{3}{2}, z^2\right) = (-1)^n \frac{n!}{(2n+1)!2} H_{2n+1}(z). \tag{4.11}$$

where H_n are the Hermite polynomials.

4.4 CHARACTERIZATION OF SELF-ADJOINT REALIZATIONS OF τ ON $[0, \infty)$

Our objective is to characterize all self-adjoint realization of τ, and by extension, of the realization of the OU operator $\mathcal{A}$ on the metric star graph $\boldsymbol{\mathcal{S}_1} \equiv [0, \infty)$. To this end, we first classify the behavior of τ at the endpoints 0 and ∞ as either *limit point* or *limit circle*, employing the *Weyl alternative* (cf. Theorem B.6 in the appendix).

Proposition 4.3 *For the* SL *differential expression* $\tau f = \frac{-\left(\frac{1}{2}e^{-x^2} f'\right)'}{e^{-x^2}}$ *on* $[0, \infty)$*, the endpoint* 0 *is in the limit circle case, and the endpoint* ∞ *is in the limit point case for all* $\lambda \in \mathbb{C}$*.*

This means that for τ, all solutions of $(\tau - \lambda)f = 0$ are $L^2_{\gamma_1}$–integrable near 0 (limit circle case at 0), but there exists at least one solution that is not $L^2_{\gamma_1}$–integrable near ∞ for each $\lambda \in \mathbb{C}$ (limit point case at ∞).

Proof By Lemma 4.2, a fundamental pair of solutions to the equation $\tau f = \lambda f$ is given by

$$M\left(-\frac{\lambda}{2}, \frac{1}{2}, x^2\right) \quad \text{and} \quad xM\left(\frac{1-\lambda}{2}, \frac{3}{2}, x^2\right),$$

where M is the Kummer function.

For the specific case of $\lambda = 0$, these solutions simplify to:

$$\theta_1(x) := M\left(0, \frac{1}{2}, x^2\right) = 1,$$

$$\theta_2(x) := xM\left(\frac{1}{2}, \frac{3}{2}, x^2\right) = \frac{\sqrt{\pi}}{2} \operatorname{erfi}(x),$$

where $\operatorname{erfi}(x)$ is the imaginary error function.[3]

Now, let's analyze the integrability of θ_1 and θ_2 near zero and infinity in $L^2_{\gamma_1}(0, \infty)$:

- $\boldsymbol{\theta_1}$: $\int_0^\infty |\theta_1(x)|^2 \gamma_1(x) \mathrm{d}x = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \mathrm{d}x = 1 < \infty$. So, θ_1 is integrable near zero and infinity, and thus lies left and right in $L^2_{\gamma_1}(0, \infty)$ (see Definition B.4).

- $\boldsymbol{\theta_2}$: As $x \to 0$, $\operatorname{erfi}(x) \sim x$, so $\int_0^c |\theta_2(x)|^2 \gamma_1(x) \mathrm{d}x < \infty$ for some $c > 0$. Thus, θ_2 is integrable near zero and lies left in $L^2_{\gamma_1}(0, \infty)$. However, as $x \to \infty$, $\operatorname{erfi}(x) \sim \frac{e^{x^2}}{\sqrt{\pi} x}$, so $\int_c^\infty |\theta_2(x)|^2 \gamma_1(x) \mathrm{d}x = \infty$ for any $c > 0$. Hence, θ_2 is not integrable near infinity and does not lie right in $L^2_{\gamma_1}(0, \infty)$.

Since both θ_1 and θ_2 lie left in $L^2_{\gamma_1}(0, \infty)$ at 0, and only θ_1 lies right in $L^2_{\gamma_1}(0, \infty)$ for $\lambda = 0$, the endpoint 0 is in the limit circle case and the endpoint ∞ is in the limit

3 Note that $\frac{d}{dz} \operatorname{erfi} z = \frac{2}{\sqrt{\pi}} e^{z^2}$.

point case for $\lambda = 0$. By Theorem B.5 in the appendix, this classification extends to all $\lambda \in \mathbb{C}$, completing the proof. ∎

By Proposition 4.3, the SL expression τ on $[0, \infty)$ has a limit circle case at 0 and a limit point case at ∞ for all $\lambda \in \mathbb{C}$.

Proposition 4.4 *All self-adjoint restrictions of the maximal operator T, or equivalently, all self-adjoint realizations of the* SL *differential expression τ on $[0, \infty)$, are characterized by the boundary condition*

$$(\sin\delta) f'(0) + (\cos\delta) f(0) = 0, \quad \text{for } f \in \operatorname{Dom}(T), \tag{4.12}$$

where $\delta \in [0, \pi)$.

Proof According to [HK92, Case 7.1], all self-adjoint extensions are characterized by the boundary condition

$$\alpha \lim_{x \to 0} W[f, \theta_1] + \beta \lim_{x \to 0} W[f, \theta_2] = 0$$

where $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ satisfy $\alpha\overline{\beta} \in \mathbb{R}$, and $W[u_1, u_2]$ denotes the *modified Wronskian* of two solutions u_1 and u_2, as defined in Definition B.3 in the appendix.

We have

$$\alpha \lim_{x \to 0} W[f, \theta_1] + \beta \lim_{x \to 0} W[f, \theta_2] = -\frac{\alpha}{2} f'(0) + \frac{\beta}{2} f(0) = 0. \tag{4.13}$$

Multiplying equation (4.13) by a non-zero complex constant allows us to make α real without loss of generality. The condition $\alpha\overline{\beta} = \beta\overline{\alpha}$ then ensures that β is also real.

Normalizing equation (4.13) by $\frac{1}{2}\sqrt{\alpha^2 + \beta^2}$ and defining $\delta \in [0, \pi)$ such that

$$\sin\delta = \frac{-\alpha}{\sqrt{\alpha^2 + \beta^2}}$$

we obtain the boundary condition (4.12). ∎

4.5 EIGENVALUES AND EIGENFUNCTIONS FOR SPECIAL CASES OF δ

Having characterised the self-adjoint realizations of τ, we now construct explicit solutions to the eigenvalue equation that satisfy the corresponding Boundary Conditions (BCs) (4.12). By (4.1), these solutions are eigenfunctions of the self-adjoint realizations of the OU operator $\mathcal{A}$ in $\operatorname{Dom}(T)$, with eigenvalues of opposite sign.

Proposition 4.5 *For $\delta \in [0, \pi)$, let $f_\delta : [0, \infty) \to \mathbb{R}$ be defined by*

$$f_\delta(x) := (\cos\delta) x M\left(\frac{1-\lambda}{2}, \frac{3}{2}, x^2\right) - (\sin\delta) M\left(-\frac{\lambda}{2}, \frac{1}{2}, x^2\right), \tag{4.14}$$

where M denotes the Kummer function. Then f_δ is a solution to the eigenvalue equation

$$\tau f = \lambda f \tag{4.15}$$

on $[0, \infty)$*, satisfying the boundary condition*

$$(\sin \delta) f'(0) + (\cos \delta) f(0) = 0. \tag{4.16}$$

Proof By Lemma 4.2, $M(-\frac{\lambda}{2}, \frac{1}{2}, x^2)$ and $xM(\frac{1-\lambda}{2}, \frac{3}{2}, x^2)$ form a fundamental pair of solutions to equation (4.15). Therefore, f_δ, being a linear combination of these solutions, satisfies (4.15). Direct calculation verifies that f_δ also satisfies the boundary condition (4.16). ∎

Remark 4.6 *It is important to note that* f_δ *as defined in Proposition 4.5 does not generally belong to* $\mathrm{Dom}(T)$*. The values of* λ *for which* $f_\delta \in \mathrm{Dom}(T)$ *constitute the eigenvalues of the corresponding self-adjoint realisation of* τ*.*

To determine these eigenvalues explicitly, we now examine f_δ for specific values of δ, corresponding to classical BCs. For each case, we analyse the asymptotic behaviour of f_δ to ascertain when it belongs to $\mathrm{Dom}(T)$, thus identifying the eigenvalues and associated eigenfunctions.

Let's consider the following combinations:

- $\boldsymbol{\delta = 0}$ (Dirichlet BCs) The solution is given by $xM\left(\frac{1-\lambda}{2}, \frac{3}{2}, x^2\right)$. From the asymptotic analysis (4.10), we can see that the function belongs to $L^2_{\gamma_1}(0, \infty)$ for $\lambda = 2n + 1, n \in \mathbb{Z}^{\geq 0}$. Further, we have,

$$xM\left(-n, \frac{3}{2}, x^2\right) = (-1)^n \frac{n!}{(2n+1)!} H_{2n+1}(x), \tag{4.17}$$

  where $H_{2n+1}(x)$ are the Hermite polynomials of odd order.

  We have thus as eigenfunctions $H_{2n+1}(x)$ and eigenvalues $2n + 1$ for $n \in \mathbb{Z}^{\geq 0}$.

- $\boldsymbol{\delta = \frac{\pi}{2}}$ (Neumann BCs) The solution is given by $-M\left(-\frac{\lambda}{2}, \frac{1}{2}, x^2\right)$. From the asymptotic analysis (4.11), we see that the function belongs to $L^2_{\gamma_1}(0, \infty)$ for $\lambda = 2n, n \in \mathbb{Z}^{\geq 0}$. Further, we have,

$$M\left(-n, \frac{1}{2}, x^2\right) = (-1)^n \frac{m!}{(2n)!} H_{2n}(x), \tag{4.18}$$

  where $H_{2n}(x)$ are the Hermite polynomials of even order.

  We have thus as eigenfunctions $H_{2n}(x)$ and eigenvalues $2n$ for $n \in \mathbb{Z}^{\geq 0}$.

- $\boldsymbol{\delta \in (0, \pi/2) \cup (\pi/2, \pi)}$ (Robin BCs) The solution is given by (4.14).

  We have to find λ such that f_δ is in $L^2_{\gamma_1}(0, \infty)$.

  From the asymptotic analysis in (4.8), we have

$$\begin{aligned} f_\delta(x) \sim e^{x^2}(x^2)^{-\frac{\lambda}{2}-\frac{1}{2}} &\left((\cos \delta) \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1-\lambda}{2})} - (\sin \delta) \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\lambda}{2})} \right) \\ &+ \mathcal{O}(x^{-2}), \quad \text{as } x \to \infty. \end{aligned} \tag{4.19}$$

  Thus, the eigenvalues of the corresponding τ realization are given by the intersection of the plot of $\tan \delta$ with the plot of $\frac{\Gamma(\frac{3}{2})\Gamma(\frac{-\lambda}{2})}{\Gamma(\frac{1-\lambda}{2})\Gamma(\frac{1}{2})} = \frac{\Gamma(\frac{-\lambda}{2})}{2\Gamma(\frac{1-\lambda}{2})}$.

See Figure 4.2 for the case $\tan\delta = 2$. Note that for $\delta \in \left(0, \frac{\pi}{2}\right)$, there exists a negative eigenvalue.

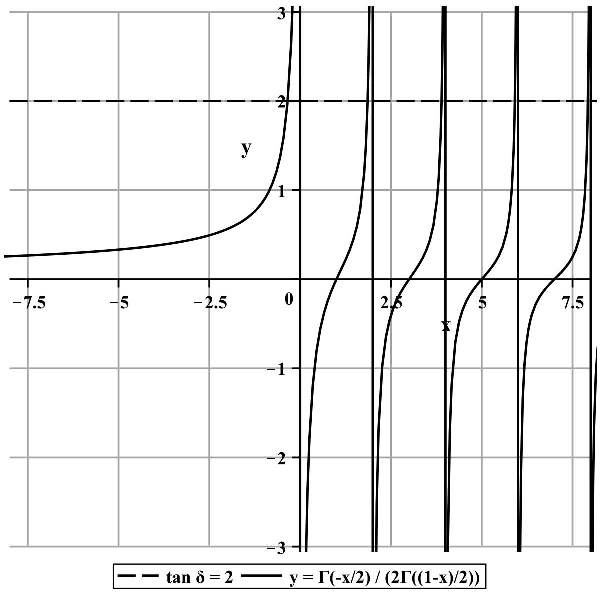

Figure 4.2: τ-Eigenvalues for Robin BC with $\tan\delta = 2$.

Building upon our characterization of the self-adjoint realizations of τ, we now present a corresponding result for the OU operator $\mathcal{A}$. This proposition extends [MR22, Lemma 4.7] to include Robin BCs, providing a comprehensive spectral characterization. [4]

Proposition 4.7 *Let $\delta \in (0, \pi/2) \cup (\pi/2, \pi)$ and consider the operator A_δ, representing the realization of $\mathcal{A}$ on* $\mathrm{Dom}(T)$ *with Robin BCs*

$$(\sin\delta) f'(0) + (\cos\delta) f(0) = 0. \tag{4.20}$$

Then:

1. *The spectrum of A_δ is purely discrete.*
2. *The eigenvalues $\lambda \in \mathbb{R}$ of A_δ satisfy*
$$\frac{\Gamma(\frac{\lambda}{2})}{2\Gamma(\frac{1+\lambda}{2}))} = \tan\delta, \tag{4.21}$$
where $\Gamma(\cdot)$ denotes the Gamma function.
3. *The corresponding eigenfunctions are given by*
$$f_\delta(x) = (\cos\delta) x M\left(\frac{1+\lambda}{2}, \frac{3}{2}, x^2\right) - (\sin\delta) M\left(\frac{\lambda}{2}, \frac{1}{2}, x^2\right). \tag{4.22}$$
4. *For $\delta \in \left(0, \frac{\pi}{2}\right)$, A_δ admits one positive eigenvalue.*

4 Recall that by (4.1), the eigenvalues of $\mathcal{A}$ are the negatives of those for τ.

Proof The proof follows directly from our analysis of τ and the relation $\mathcal{A} = -\tau$ established in (4.1). The spectral properties of A_δ are thus mirror images of those for the corresponding realization of τ, with the sign of the eigenvalues reversed. ∎

This extension of [MR22, Lemma 4.7] provides a characterisation of the spectrum of the OU operator on $\mathrm{Dom}(T)$ under Robin BCs, generalising the Dirichlet and Neumann cases presented in the original lemma.

4.6 EIGENVALUES AND EIGENFUNCTIONS OF $\mathcal{A}$ ON $\boldsymbol{\mathcal{S}_m}$

We now extend our analysis to the operator $\mathcal{A}$ on $\boldsymbol{\mathcal{S}_m}$. Recall that $\boldsymbol{r_j}$, $j \in I$,[5] denote the rays of the metric graph, and $\mathbf{0}$ its central vertex. For a function $\boldsymbol{f} : \boldsymbol{\mathcal{S}_m} \to \mathbb{K}$, we denote by f_j its restriction to the ray $\boldsymbol{r_j}$.[6]

On each ray $\boldsymbol{r_j}$, we consider the OU differential expression given by

$$\mathcal{A} f_j = -\tau f_j = \frac{1}{2}\frac{d^2 f_j}{dx^2} - y\frac{d f_j}{dx}, \quad x \in [0, \infty). \tag{4.23}$$

To characterize specific self-adjoint realizations of the OU operator on $\boldsymbol{\mathcal{S}_m}$, we employ the Hermitian VCs established in Theorem 2.20.

We focus on replicating and extending the results of [MR22] by considering three specific cases of VCs while ensuring continuity at the vertex $\mathbf{0}$:

- Dirichlet-Kirchhoff VCs: $\sum_{j\in I} f_j'(\mathbf{0}_j) = 0$, $\boldsymbol{f}(\mathbf{0}) = 0$
- Neumann-Kirchhoff VCs: $\sum_{j\in I} f_j'(\mathbf{0}_j) = 0, \boldsymbol{f}(\mathbf{0}) \neq 0$
- δ-Coupling VCs: $\sum_{j\in I} f_j'(\mathbf{0}_j) = c\boldsymbol{f}(\mathbf{0})$, where $c \in \mathbb{R} \setminus \{0\}$ and $\boldsymbol{f}(\mathbf{0}) \neq 0$.

Additionally, we introduce the concept of *strict Neumann-Kirchhoff* VCs:

- Strict Neumann-Kirchhoff VCs: $f_j'(\mathbf{0}_j) = 0$ for all $j \in I$ and $\boldsymbol{f}(\mathbf{0}) \neq 0$.

The relationship between Neumann-Kirchhoff and strict Neumann-Kirchhoff conditions will be explored in subsequent analysis.

Synthesizing the SL analysis from Section 4.1 with the single-ray case results of Section 4.4, we now characterize the spectrum and eigenfunctions for these cases on the metric star graph $\boldsymbol{\mathcal{S}_m}$.

4.6.1 *Dirichlet- and strict Neumann-Kirchhoff VCs.*

We will first characterize the spectrum and eigenfunctions for the Dirichlet-Kirchhoff and strict Neumann-Kirchhoff VCs. We will prove in Proposition 4.9 that strict and standard Neumann-Kirchhoff VCs are equivalent.

5 In this thesis $I := \{1, \ldots, m\}$.
6 In this section, we reserve the letter i for the imaginary unit, i.e., $i^2 = -1$.

Proposition 4.8 (Dirichlet- and strict Neumann-Kirchhoff cases) *Let $\boldsymbol{A_2}$ denote the operator obtained by restricting the OU operator $\mathcal{A}$ in $L^2_{\gamma_m}(\boldsymbol{\mathcal{S}_m})$ with domain*

$$\begin{aligned}\operatorname{Dom}(\boldsymbol{A_2}) = \{\boldsymbol{f} \in L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m}) : f_j, f_j' \text{ are absolutely continuous on } \boldsymbol{r}_j \\ \text{for each } j \in I,\ B\boldsymbol{f}(\mathbf{0}) = C\boldsymbol{f}'(\mathbf{0}),\ \mathcal{A}\boldsymbol{f} \in L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m})\},\end{aligned}$$

where B and C are $m \times m$ matrices given by

$$B = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

These matrices encode continuity at the vertex $\mathbf{0}$ and the Kirchhoff condition $\sum_{j\in I}^{m} \boldsymbol{f}_j'(\mathbf{0}) = 0$.

Then:

(i) For Dirichlet-Kirchhoff VCs the spectrum of $\boldsymbol{A_2}$ is given by

$$\sigma(\boldsymbol{A_2}) = \{-(2n+1) : n \in \mathbb{N}_0\},$$

where each eigenvalue has multiplicity $m-1$. The corresponding eigenfunctions are of the form

$$\boldsymbol{f}_{n,l}(\boldsymbol{x}_j) = \alpha_{j,l} \frac{(-1)^n n!}{(2n+1)!} H_{2n+1}(|\boldsymbol{x}_j|)$$

where H_{2n+1} are the odd Hermite polynomials, and the coefficients $\alpha_{j,l}$ are defined as:

$$\alpha_{j,l} := \begin{cases} e^{(2\pi(j-1)/l)i}, & \text{for } 1 \le j \le l, \\ 0, & \text{for } l+1 \le j \le m. \end{cases}$$

where $j \in I$ and $l \in \{2, \dots, m\}$.

(ii) For strict Neumann-Kirchhoff VCs $\left(f_j'(\mathbf{0}_j) = 0 \text{ for all } j \in I \text{ and } \boldsymbol{f}(\mathbf{0}) \neq 0\right)$, the spectrum of $\boldsymbol{A_2}$ is given by

$$\sigma(\boldsymbol{A_2}) = \{-2n : n \in \mathbb{N}_0\},$$

where each eigenvalue has multiplicity 1. The corresponding eigenfunctions are of the form

$$\boldsymbol{f}_n(\boldsymbol{x}_j) = (-1)^n \frac{\beta n!}{(2n)!} H_{2n}(|\boldsymbol{x}_j|),$$

where H_{2n} are the even Hermite polynomials and $\beta := \boldsymbol{f}(\mathbf{0})$.

Proof First, note that the matrices B and C satisfy condition (2.26) $BC^* = CB^* = 0$, ensuring that $\boldsymbol{A_2}$ is self-adjoint. We consider the SL problem $\frac{1}{2}\frac{d^2 f_j}{dx^2} - x\frac{df_j}{dx} = \lambda f_j$ on each ray $\boldsymbol{r_j}$.

(i) For Dirichlet conditions, the Dirichlet case in Proposition 4.7 yields the solutions $f_j(x) = \alpha_j x M(\frac{1+\lambda}{2}, \frac{3}{2}, x^2)$,[7] where $\alpha_j \in \mathbb{C}$ and M is the Kummer function. For f_j to be in $L^2_{\gamma_m}(0,\infty)$, we must have $\lambda = -(2n+1)$ for $n \in \mathbb{N}_0$ (see 4.11), corresponding to odd Hermite polynomials $H_{2n+1}(x)$.

A notable feature of the multi-ray case is the possibility of having rays j with the trivial solution $f_j(x) = 0$, which satisfies the boundary condition, provided at least one ray has a non-trivial solution. However, due to the Kirchhoff Vertex Condition $\sum_{j\in I} f_j'(\mathbf{0}_j) = \sum_{j\in I} \alpha_j = 0$, a single non-zero ray cannot satisfy the VC. Thus, at least two and at most m of the solutions at each ray must be non-trivial.

To construct eigenfunctions that also satisfy the Kirchhoff condition, we exploit the rotational symmetry of the star graph. Following the eigenspaces of the rotation operator $\boldsymbol{R}$ given in (3.4.2), we define the coefficients $\alpha_{j,l}$ as

$$\alpha_{j,l} := \begin{cases} e^{(2\pi(j-1)/l)i}, & \text{for } 1 \le j \le l, \\ 0, & \text{for } l+1 \le j \le m, \end{cases}$$

where $j \in I$ and $l \in \{2, \dots, m\}$. These coefficients satisfy the Kirchhoff condition, as shown below. The $m-1$ eigenfunctions corresponding to eigenvalue $-(2n+1)$ are then given by

$$f_{n,l}(\boldsymbol{x}_j) = (-1)^n \frac{n!}{(2n+1)!}\Big(\alpha_{j,l} H_{2n+1}(\boldsymbol{x}_j)\Big),$$

for each $n \in \mathbb{N}_0$, $j \in I$ and $l \in \{2, \dots, m\}$,

The coefficients $\alpha_{j,l}$ satisfy the Kirchhoff condition for $l \in \{2, \dots, m\}$, since

$$\sum_{j\in I} \alpha_{j,l} = \sum_{j=1}^{l} e^{(2\pi(j-1)/l)i} = \frac{1 - e^{(2\pi l/l)i}}{1 - e^{(2\pi/l)i}} = \frac{1-1}{1 - e^{(2\pi/l)i}} = 0.$$

Consequently, for each eigenvalue $\lambda = -(2n+1)$, we have $m-1$ linearly independent eigenfunctions, yielding a multiplicity of $m-1$.

(ii) For Neumann conditions, the Neumann case in Lemma 4.7 yields the solutions $f_i(x) = \beta_i M(\frac{\lambda}{2}, \frac{1}{2}, x^2)$. For $f_j \in L^2_{\gamma_m}(0,\infty)$, we must have $\lambda = -2n$ for $n \in \mathbb{N}_0$, corresponding to even Hermite polynomials $H_{2n}(x)$. The continuity at the central vertex implies $\beta_j = \beta_k =: \beta$ for all j,k. Hence, the eigenfunction must be the same on all rays, yielding multiplicity 1 for each eigenvalue. Note that in this case, the Kirchhoff condition $\sum_{j\in I} f_j'(\mathbf{0}_j) = 0$ is automatically satisfied. Completing the proof. This completes the characterization of the spectrum and eigenfunctions for both Dirichlet-Kirchhoff and strict Neumann-Kirchhoff VCs. ∎

The following proposition establishes the equivalence between the strict Neumann-Kirchhoff and Neumann-Kirchhoff VCs for the Ornstein-Uhlenbeck operator on the metric star graph $\boldsymbol{\mathcal{S}_m}$ with the given matrices B and C. This result justifies the focus on the strict Neumann-Kirchhoff conditions in the previous proposition.

Proposition 4.9 *For the OU operator $\boldsymbol{\mathcal{A}}$ on the metric star graph $\boldsymbol{\mathcal{S}_m}$ with matrices B and C as defined in Lemma 4.8, which encode the Standard VCs, the strict Neumann-*

7 Recall that by equation (4.1), the eigenvalues of $\mathcal{A}$ are the negatives of those for τ.

Kirchhoff VCs $\left(f_j'(\mathbf{0}) = 0 \text{ for all } j \in I,\ \boldsymbol{f}(\mathbf{0}) \neq 0\right)$ *and the Neumann-Kirchhoff* VCs $\left(\sum_{j\in I} f_j'(\mathbf{0}_j) = 0 \text{ and } \boldsymbol{f}(\mathbf{0}) \neq 0\right)$ *are equivalent.*

Proof Proposition 4.8 covers the Dirichlet-Kirchhoff case (case (i), where $\boldsymbol{f}(\mathbf{0}) = 0$) and the strict Neumann-Kirchhoff case (case (ii), where $f_j'(\mathbf{0}_j) = 0$ for all $j \in I$ and $\boldsymbol{f}(\mathbf{0}) \neq 0$). We now investigate whether there could be another case that satisfies continuity and Kirchhoff conditions, for which there exists at least one ray $\boldsymbol{r_j}$ such that $\alpha_j := f_j'(\mathbf{0}_j)$ and $\beta := f_j(\mathbf{0}_j)$ are both non-zero.

From the asymptotic analysis in Section 4.4, we know that for the ray $\boldsymbol{r_j}$, the eigenvalues are determined by the intercept of a certain function with $-\beta/\alpha_j$. For the operator to have well-defined, real eigenvalues common to all rays, these intercepts must coincide and be real-valued. This requirement implies that $-\beta/\alpha_j = -\beta/\alpha_k$ for all k, or equivalently, $\alpha_j = \alpha_k =: \alpha$ for all $j, k \in I$.

Consequently, we obtain the condition

$$\sum_{j\in I} f_j'(\mathbf{0}_j) = m\alpha \neq 0.$$

This condition contradicts the Kirchhoff condition $\sum_{j\in I} f_i'(\mathbf{0}_j) = 0$, which is encoded by the matrices B and C as defined in Lemma 4.8.

We conclude that the Neumann-Kirchhoff VCs are satisfied if and only if the strict Neumann-Kirchhoff VCs hold. ∎

4.6.2 *δ-Coupling VCs.*

Having characterised the spectrum and eigenfunctions for the Dirichlet-Kirchhoff and Neumann-Kirchhoff VCs, which represent the extreme scenarios, we now investigate the more general case of δ-Coupling VCs.

Proposition 4.10 (δ-Coupling VCs) *Let $\boldsymbol{A_2}$ be the realization of the OU operator $\mathcal{A}$ in $L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m})$ whose domain is given by*

$$\begin{aligned}\mathrm{Dom}(\boldsymbol{A_2}) = \{\boldsymbol{f} \in L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m}) : f_j, f_j' \text{ are absolutely continuous on } \boldsymbol{r}_j \\ \text{for each } j \in I,\ B\boldsymbol{f}(\mathbf{0}) = C\boldsymbol{f}'(\mathbf{0}),\ \mathcal{A}\boldsymbol{f} \in L^2_{\boldsymbol{\gamma_m}}(\boldsymbol{\mathcal{S}_m})\},\end{aligned}$$

where B and C are $m \times m$ matrices given by

$$B = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1/c & 1/c & 1/c & \cdots & 1/c \end{pmatrix},$$

where $c \in \mathbb{R} \setminus \{0\}$.

These matrices encode continuity at $\mathbf{0}$ and the δ-Coupling VC $\sum_{j\in I} f_j'(\mathbf{0}_j) = c\boldsymbol{f}(\mathbf{0})$.

We have:

(i) The spectrum of $\boldsymbol{A_2}$ on $\boldsymbol{\mathcal{S}_m}$ consists of eigenvalues λ that satisfy the equation

$$\frac{\Gamma(\frac{\lambda}{2})}{2\Gamma(\frac{1+\lambda}{2})} = -\frac{m}{c}.$$

(ii) Each eigenvalue λ has multiplicity 1, and the corresponding eigenfunction $\boldsymbol{f}_\lambda$ are given by

$$\boldsymbol{f}_\lambda(\boldsymbol{x}_j) := f_c(|\boldsymbol{x}_j|),$$

where

$$f_c(x) = \frac{c\beta}{m} xM\left(1 + \frac{\lambda}{2}, \frac{3}{2}, x^2\right) + \beta M\left(\frac{\lambda}{2}, \frac{1}{2}, x^2\right),$$

$\beta := \boldsymbol{f}(\boldsymbol{0})$ and M denotes the Kummer function of the first kind.

(iii) The spectrum consists of countably many simple eigenvalues, accumulating at $-\infty$. For $c < 0$, there exists one positive eigenvalue.

Proof First, we verify that B and C satisfy condition (2.26) $BC^* = CB^*$, ensuring that $\boldsymbol{A_2}$ is self-adjoint:

$$BC^* = CB^* = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1/c \end{pmatrix}.$$

We consider the SL problem $\frac{1}{2}\frac{d^2 f_j}{dx^2} - x\frac{df_j}{dx} = \lambda f_j$ on each ray $\boldsymbol{r}_j$.
From Proposition 4.7 the solutions on ray $\boldsymbol{r}_j$ is of the form

$$f_j(x) = \alpha_j xM\left(1 + \frac{\lambda}{2}, \frac{3}{2}, x^2\right) + \beta_j M\left(\frac{\lambda}{2}, \frac{1}{2}, x^2\right).$$

for $\alpha_j, \beta_j \in \mathbb{C}$. The continuity at the central vertex implies $\beta_j = \beta_k =: \beta$ for all j, k. From the asymptotic analysis in Section 4.4, we know that $f_j \in L_{\gamma_m}(\mathbb{R}^+)$ for the ray $\boldsymbol{r_j}$ if λ fulfils this equation:

$$\frac{\Gamma(\frac{\lambda}{2})}{\Gamma(\frac{1+\lambda}{2})} = -\frac{\beta}{\alpha_j}.$$

For the operator to have well-defined, real eigenvalues common to all rays, these intercepts must coincide and be real-valued. This implies that $-\beta/\alpha_j = -\beta/\alpha_k$ for all k, or equivalently, $\alpha_j = \alpha_k =: \alpha$ for all $j, k \in I$.

Consequently, we obtain the condition

$$\sum_{j \in I} f_j'(\boldsymbol{0}_j) = m\alpha = c\boldsymbol{f}(\boldsymbol{0}) = c\beta.$$

Thus $\alpha = \frac{c\beta}{m}$. The eigenvalue equation follows from Proposition 4.7 since

$$\frac{\Gamma(\frac{\lambda}{2})}{2\Gamma(\frac{1+\lambda}{2})} = -\frac{\beta}{\alpha_j} = -\frac{\beta}{\alpha} = -\frac{m}{c}.$$

And the rest of results follow from Proposition 4.7 for $\delta \in (0, \pi) \setminus \{\pi/2\}$, such that $\tan\delta = -\frac{m}{c}$. ∎

4.7 CONCLUSION

By establishing a connection between the Ornstein-Uhlenbeck operator $\mathcal{A}$ on $\boldsymbol{\mathcal{S}_m}$ and a singular Sturm-Liouville problem, we have derived explicit expressions for the eigenfunctions and eigenvalues of the self-adjoint realisations of $\mathcal{A}$ corresponding to Standard and δ-Coupling VCs. Specifically, we have characterised the spectrum and eigenfunctions for the operators $\boldsymbol{A_2}$ on $L^2_{\gamma_m}(\boldsymbol{\mathcal{S}_m})$ under Dirichlet-Kirchhoff, Neumann-Kirchhoff, and δ-Coupling conditions at the central vertex.

In the next chapter, we will explore another approach to extend the results of [MR22], focusing on the parabolic problem associated with the OU operator. By employing commuting operators and introducing the Extended Algebra, we will develop techniques for constructing solutions that satisfy various BCs, particularly emphasising the case of $\boldsymbol{\mathcal{S}_1}$ and Robin BCs.

5 COMMUTING OPERATORS: SOLVING THE PARABOLIC PROBLEM (P_A) ON $\mathbb{R}$ WITH ROBIN BOUNDARY CONDITIONS

"The most frightening phrase in the Russian language is 'That's odd.'"
— Isaac Asimov [Asi87]

5.1 INTRODUCTION AND MOTIVATION

Our goal is to extend [MR22, Theorem 3.3] to δ-Coupling Vertex Conditions (VCs) of the form $\sum_{i\in I} f_i'(\mathbf{0}_i) = h\boldsymbol{f}(0)$, where h is the coupling parameter. As a first step towards this objective, we focus on the case $m = 1$, which corresponds to Robin Boundary Conditions (BCs) on the half-line $[0, \infty)$, leaving the generalization to arbitrary m for future work.

Our approach is inspired by the classical treatment of the heat equation on a semi-infinite interval with Robin BCs, as studied by Bryan [Bry91]. Bryan's method, which exploits symmetry properties and spatial derivative behavior, can be viewed as another application of the even-odd extension technique, akin to the approach employed in Lemma 3.4. By adapting this technique to the Ornstein-Uhlenbeck (OU) operator, we construct solutions to the parabolic problem (P_A) with Robin BCs on $[0, \infty)$. This setting represents the simplest non-trivial metric star graph $\boldsymbol{\mathcal{S}_1}$.

The key insight is that if $u(x,t)$ is a solution to the heat equation, then so is $\partial_x u(x,t)$. [Bry91] exploits this property by considering a linear combination of u and $\partial_x u$, specifically designed to satisfy the Robin BC at $x = 0$. By extending the problem to the whole real line and imposing appropriate symmetry conditions on the initial data, Bryan ensures that this linear combination naturally satisfies the desired boundary condition when restricted back to the semi-infinite interval. This method works elegantly for the heat equation because:

1. The spatial derivative operator ∂_x commutes with both the time derivative ∂_t and the spatial Laplacian ∂_x^2.
2. The heat equation is invariant under spatial reflection $x \mapsto -x$.
3. Linear combinations of solutions to the heat equation are also solutions.

These properties allow for the construction of solutions with specific symmetries that satisfy the desired BCs when restricted to the original domain.

The OU operator $\mathcal{A}$, however, presents a fundamentally different situation. Unlike the heat equation, where ∂_x commutes with the spatial part of the operator, the OU operator exhibits non-trivial commutation relations with spatial derivatives. Specifically, while $[\mathcal{A}, \partial_t] = 0$ still holds, we have $[\mathcal{A}, \partial] = \partial$ and $[\mathcal{A}, \partial^*] = -\partial^*$, where ∂ and ∂^* are the annihilation and creation operators defined in (2.13). These non-zero commutators mean that if $u(x,t)$ is a solution to the OU equation, ∂u and $\partial^* u$ are generally not solutions. This non-commutativity prevents us from directly applying Bryan's method or similar classical approaches based on spatial derivatives of solutions.

5.2 CHARACTERIZATION OF COMMUTING OPERATORS

To overcome this challenge, we introduce the OU *evolution operator* $\square_{\mathcal{A}}$, defined as:

$$\square_{\mathcal{A}} := \partial_t - \mathcal{A}, \tag{5.1}$$

where $\mathcal{A}$ is the OU operator. Our strategy is to find operators that commute with $\square_{\mathcal{A}}$, allowing us to generate new solutions that accommodate different BCs. Before proceeding, let us recall the definition of the commutator:

Definition 5.1 *Let P and Q be two linear operators with domains* $\mathrm{Dom}(P)$ *and* $\mathrm{Dom}(Q)$*, respectively. The commutator of P and Q is defined as*

$$[P, Q] := PQ - QP, \tag{5.2}$$

with domain $\mathrm{Dom}([P,Q]) = \mathrm{Dom}(PQ) \cap \mathrm{Dom}(QP)$*, where*

$$\mathrm{Dom}(PQ) = \{f \in \mathrm{Dom}(Q) : Qf \in \mathrm{Dom}(P)\},$$
$$\mathrm{Dom}(QP) = \{f \in \mathrm{Dom}(P) : Pf \in \mathrm{Dom}(Q)\}.$$

If $[P,Q] = 0$ *on* $\mathrm{Dom}([P,Q])$*, then the operators P and Q are said to commute.*

To characterize operators that commute with $\square_{\mathcal{A}}$, we consider a suitable function space:

Definition 5.2 *Let $\nu \in (0,1), q \in \mathbb{N}$. We define the function space $\mathcal{D}_\nu^q$ as*

$$\mathcal{D}_\nu^q := C([0,\infty) \times \mathbb{R}) \cap C_{loc}^{q+\frac{\nu}{2},1+q+\nu}((0,\infty) \times \mathbb{R}).$$

The following proposition characterizes the most general first-order differential operator on $\mathcal{D}_\nu^q$ that commutes with $\square_{\mathcal{A}}$:

Proposition 5.3 *Let $H_\Omega : \mathcal{D}_\nu^2 \to \mathcal{D}_\nu^1$ defined by*

$$\begin{aligned}(H_\Omega u)(x,t) &:= \omega_{000} u(x,t) - \omega_{100} e^{-t} \partial^* u(x,t) + \omega_{010} e^t \partial u(x,t) + \omega_{001}\, \partial_t u(x,t) \\ &= \left(\omega_{000} - 2\omega_{100} x e^{-t}\right) u(x,t) + \left(\omega_{100} e^{-t} + \omega_{010} e^t\right) \frac{\partial u(x,t)}{\partial x}) + \omega_{001} \frac{\partial u(x,t)}{\partial t},\end{aligned} \tag{5.3}$$

where $\Omega := \{\omega_{000}, \omega_{100}, \omega_{010}, \omega_{001}\} \subset \mathbb{K}$ and ∂^*,∂ are the creation and annihilation operators defined in (2.13)*.*

Then, for any $u \in \mathcal{D}_\nu^2$, the operator H_Ω commutes with the OU *evolution operator $\square_{\mathcal{A}}$, i.e.,*

$$[\square_{\mathcal{A}}, H_\Omega] u = 0, \quad \forall u \in \mathcal{D}_\nu^2. \tag{5.4}$$

Moreover, any first-order differential operator that commutes with $\square_{\mathcal{A}}$ on $\mathcal{D}_\nu^2$ is of the form H_Ω for some choice of the constants $\omega_{000}, \omega_{100}, \omega_{010}, \omega_{001} \in \mathbb{K}$.

Proof The commutation property $[\square_{\mathcal{A}}, H_\Omega] = 0$ follows from a direct computation:

$$\begin{aligned}(\square_{\mathcal{A}}(H_\Omega u))(x,t) &= -\frac{1}{2}(\omega_{100}e^{-t} + \omega_{010}e^{t})\frac{\partial^3 u(x,t)}{\partial x^3} - \frac{\omega_{001}}{2}\frac{\partial^3 u(x,t)}{\partial t \partial x^2} + \omega_{001}\frac{\partial^2 u(x,t)}{\partial t^2} \\ &+ \frac{1}{2}(4\omega_{100}xe^{-t} + 2x\omega_{010}e^{t} - \omega_{000})\frac{\partial^2 u(x,t)}{\partial x^2} + (\omega_{001}x + \omega_{100}e^{-t} + \omega_{010}e^{t})\frac{\partial^2 u(x,t)}{\partial t \partial x} \\ &+ \left(-2\left(x^2 - \frac{1}{2}\right)\omega_{100}e^{-t} + \omega_{010}e^{t} + \omega_{000}x\right)\frac{\partial u(x,t)}{\partial x} \\ &+ \left(\omega_{000} - 2\omega_{100}xe^{-t}\right)\frac{\partial u(x,t)}{\partial t} = (H_\Omega(\square_{\mathcal{A}}u))(x,t),\end{aligned}$$

where we have taken advantage of the equality of mixed partial derivatives for $u \in \mathcal{D}^2_\nu$.

To show that any first-order differential operator commuting with $\square_{\mathcal{A}}$ is of the form H_Ω, let C be such an operator. We can write C in the general form

$$C(u)(x,t) := c_0(x,t)u(x,t) + c_1(x,t)\frac{\partial u(x,t)}{\partial x} + c_2(x,t)\frac{\partial u(x,t)}{\partial t}, \tag{5.5}$$

where c_0, c_1 and $c_2 \in \mathcal{D}^2_\nu$. The commutator $[\square_{\mathcal{A}}, C]$ applied to $u \in \mathcal{D}^2_\nu$ gives

$$\begin{aligned}[\square_{\mathcal{A}}, C]u = &-\frac{\partial c_1}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{\partial c_2}{\partial x}\frac{\partial^2 u}{\partial t \partial x} + \left(-\frac{1}{2}\frac{\partial^2 c_1}{\partial x^2} + x\frac{\partial c_1}{\partial x} - \frac{\partial c_0}{\partial x} + \frac{\partial c_1}{\partial t} - c_1\right)\frac{\partial u}{\partial x} \\ &+ \left(-\frac{1}{2}\frac{\partial^2 c_2}{\partial x^2} + x\frac{\partial c_2}{\partial x} + \frac{\partial c_2}{\partial t}\right)\frac{\partial u}{\partial t} + \left(-\frac{1}{2}\frac{\partial^2 c_0}{\partial x^2} + x\frac{\partial c_0}{\partial x} + \frac{\partial c_0}{\partial t}\right)u.\end{aligned} \tag{5.6}$$

For the commutator to vanish for arbitrary functions $u \in \mathcal{D}^2_\nu$, the coefficients of $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial t \partial x}$, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$, and u in (5.6) must vanish. From the coefficients of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t \partial x}$, we infer that c_1 and c_2 depend only on t. This leads to the following equation by requiring the coefficient of $\frac{\partial u}{\partial x}$ to be zero:

$$-\frac{\partial c_0}{\partial x} + c_1'(t) - c_1(t) = 0, \tag{5.7}$$

with solution

$$c_0(x,t) = (c_1'(t) - c_1(t))x + g(t), \tag{5.8}$$

where $g(t)$ is an arbitrary function. Substituting (5.8) into the coefficient of u in leads to the equation

$$xc_1(t) - xc_1''(t) - g'(t) = 0. \tag{5.9}$$

From (5.9), we deduce that $g(t) =: \widetilde{\omega}_{000} \in \mathbb{K}$ is a constant and $c_1(t)$ satisfies the equation

$$c_1(t) - c_1''(t) = 0, \tag{5.10}$$

with solution

$$c_1(t) = \widetilde{\omega}_{100}e^{-t} + \widetilde{\omega}_{010}e^{t}, \tag{5.11}$$

for arbitrary constants $\widetilde{\omega}_{100}, \widetilde{\omega}_{010} \in \mathbb{K}$. Substituting (5.11) into (5.8) yields

$$c_0(x,t) = \widetilde{\omega}_{000} - 2\widetilde{\omega}_{100}xe^{-t}. \tag{5.12}$$

Finally, from the coefficient of $\frac{\partial u}{\partial t}$, we infer that $c_2(t) =: \widetilde{\omega}_{001}$ is a constant function.

Comparing (5.12) and (5.11) with (5.3), we conclude that the operator C equals $H_{\widetilde{\mathcal{P}}}$, for $\widetilde{\mathcal{P}} = \{\widetilde{\omega}_{000}, \widetilde{\omega}_{100}, \widetilde{\omega}_{010}, \widetilde{\omega}_{001}\}$, completing the proof. ∎

Proposition 5.3 characterizes first-order linear differential operators commuting with $\square_{\mathcal{A}}$. While this characterization is limited to first-order operators, the following corollary applies to commuting operators of any order. This result generates new solutions to the parabolic problem and establishes a foundation for adapting Bryan's method [Bry91] to the OU evolution equation. In Chapter 6, we will further generalize the characterization of commuting operators to linear operators of arbitrary order.

Corollary 5.4 *Let $\nu \in (0,1)$ and $v \in \mathcal{D}^2_\nu$ be the unique solution to the parabolic problem*

$$\begin{cases} \square_{\mathcal{A}} v = 0, & x \in \mathbb{R}, t > 0, \\ v(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{5.13}$$

where $f \in C^1_b(\mathbb{R})$. Define $w := H_\Omega(v)$, where the operator H_Ω is given by (5.3) *and the set $\Omega = \{\omega_{ijk}\}_{i+j+k \in \{0,1\}} \subset \mathbb{K}$. Then w is the unique solution to the transformed parabolic problem*

$$\begin{cases} \square_{\mathcal{A}} w = 0, & x \in \mathbb{R}, t > 0, \\ w(x,0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{5.14}$$

where the initial condition ψ is given by

$$\psi(x) := (\omega_{000} - 2\omega_{100} x) f(x) + (\omega_{100} + \omega_{010}) f'(x) + \omega_{001} \partial_t v(x,0). \tag{5.15}$$

Proof Let $v \in \mathcal{D}^2_\nu$ be the solution to the parabolic problem 5.13. Define the function $w := H_\Omega(v)$. Since H_Ω commutes with $\square_{\mathcal{A}}$ in $\mathcal{D}^2_\nu$ (Proposition 5.3), we have:

$$\square_{\mathcal{A}} w = \square_{\mathcal{A}}(H_\Omega(v)) = H_\Omega(\square_{\mathcal{A}} v) = H_\Omega(0) = 0.$$

Then, w is a solution to the transformed parabolic problem. For the initial condition, we have:

$$\begin{aligned} w(x,0) &= (H_\Omega v)(x,t)|_{t=0} \\ &= (\omega_{000} - 2\omega_{100} x) f(x) + (\omega_{100} + \omega_{010}) f'(x) + \omega_{001} \partial_t v(x,0). \end{aligned}$$

The uniqueness of the solution w follows from [LB06, Theorem 9.1.1]. ∎

The operator H_Ω takes different forms depending on the choice of constants ω_{000}, ω_{100}, ω_{100}, and ω_{100}. Table 5.1 illustrates several examples of these operators, including some particularly relevant for constructing solutions with *time-dependent* Robin BCs.

Characterizing operators H_Ω commuting with $\square_{\mathcal{A}}$ on $\mathcal{D}^2_\nu$ provides a framework for generating new solutions from known ones. In the following section, we apply the operator from the second row of Table 5.1 to study the OU equation on $[0, \infty)$ with *time-dependent* Robin conditions.

ω_{000}	ω_{100}	ω_{100}	ω_{001}	$H_\Omega(u)(x,t)$
1	0	0	1	$u(x,t)+\partial_t u(x,t)$
1	0	-1	0	$u(x,t)-e^t\partial u(x,t)$
1	1	0	0	$u(x,t)-e^{-t}\partial^* u(x,t)$

Table 5.1: Examples of first-order differential operators H_Ω commuting with $\Box_\mathcal{A}$ for various choices of parameters ω_{ijk}.

5.3 TIME DEPENDENT ROBIN BOUNDARY CONDITIONS FOR $\Box_\mathcal{A}$ ON $[0,\infty)$.

Consider the parabolic problem for the OU evolution operator $\Box_\mathcal{A}$ on the half-line $[0,\infty)$ with *decaying* Robin BCs[1]:

$$\Box_\mathcal{A} v(t,x) = 0, \quad t>0, x\in(0,\infty) \tag{5.16}$$

$$v(0,x) = f(x), \quad x\in[0,\infty), \tag{5.17}$$

$$\partial_x v(t,0) = he^{-t}v(t,0), \quad t>0, \tag{5.18}$$

where $0\neq h\in\mathbb{R}$ is the *coupling parameter* and $f\in C^1_b(\mathbb{R})$.

To solve this problem, we will employ the commuting operator H_Ω derived in Proposition 5.3. Specifically, we choose the constants $\omega_{000} := -h$, $\omega_{100} := 0, \omega_{010} := 1$ and $\omega_{001} := 0$, which yields the operator

$$(H_\Omega u)(x,t) = -hu(x,t) + e^t\frac{\partial u(x,t)}{\partial x}. \tag{5.19}$$

We follow the steps analogous to those used by Bryan [Bry91] for the heat equation. See also [Stend; Str07].

Step 1: Extension of the problem to the real line.

To construct solutions satisfying the Robin condition (5.18), we extend the initial condition f (see 5.17) to a function $\widetilde{f}$ on $\mathbb{R}$. The invariance of $\Box_\mathcal{A}$ under $x\mapsto -x$, as used in Theorem 3.3, ensures that solutions to the extended OU evolution equation

$$\Box_\mathcal{A} v(t,x) = 0, \quad t>0, x\in\mathbb{R},, \tag{5.20}$$

$$v(0,x) = \widetilde{f}(x), \quad x\in\mathbb{R}, \tag{5.21}$$

inherit the parity of $\widetilde{f}$. This property allows us to choose $\widetilde{f}$ such that the restriction of the solution to $[0,\infty)$ satisfies (5.18).

Step 2: Introduction of an auxiliary function.

Define the auxiliary function $w := H_\Omega v = -hv + e^t\partial_x v$. By Proposition 5.3, the commutation property $[\Box_\mathcal{A}, H_\Omega] = 0$ implies that if v solves the extended OU evolution equation (5.20), then w also satisfies

$$\Box_\mathcal{A} w(t,x) = 0, \quad t>0, x\in\mathbb{R}, \tag{5.22}$$

with initial condition

1 The label *decaying* refers to the fact that the Robin BCs tends to Neumann for $t\to\infty$.

$$w(0,x) = (H_\Omega \widetilde{f})(x) = -h\widetilde{f}(x) + \widetilde{f}'(x). \tag{5.23}$$

The goal is to choose $\widetilde{f}$ such that $w(0,x)$ is an odd function, which will imply that $w(t,x)$ is odd in x for all $t > 0$.

Step 3: Construction of the extended initial condition.
To ensure that $w(0,x)$ is odd, we define the extended initial condition $\widetilde{f}$ as

$$\widetilde{f}(x) := \begin{cases} f(x), & x \geq 0, \\ g(x), & x \leq 0, \end{cases} \tag{5.24}$$

where g is determined by solving the equation

$$g'(x) - hg(x) = -f'(-x) + hf(-x) \tag{5.25}$$

with the condition $g(0) = f(0)$. The solution, adapted from [Str07, Exercise 5 in Section 3.1] and [Stend], is given by

$$g(x) = f(-x) - e^{hx}\left[2h\int_0^{-x} f(y)e^{hy}\mathrm{d}y\right], \quad x \leq 0. \tag{5.26}$$

By construction, $\widetilde{f}$ ensures that $w(0,x)$ is odd. Consequently, the invariance of $\Box_{\mathcal{A}}$ under $x \mapsto -x$ implies that $w(t,x)$ is odd in x for all $t > 0$. This oddness of $w(t,x)$ yields $\partial_x v(0,t) = e^{-t}hv(0,t)$, guaranteeing that the restriction of v to $[0,\infty)$ satisfies the Robin condition (5.18).

Step 4: Explicit Solution via Kernel Representation
Let $\widetilde{f}$ denote the extended initial condition on $\mathbb{R}$. The solution $v(x,t)$ to the OU equation on $\mathbb{R}$ admits the integral representation

$$v(t,x) = \int_{\mathbb{R}} \kappa_{OU}(t,x,y)\widetilde{f}(y)\mathrm{d}y, \tag{5.27}$$

where $\kappa_{OU}(t,x,y)$ is the OU kernel (3.24). Splitting the integral and applying the change of variables $y \mapsto -y$ in the negative half-line yields

$$\begin{aligned} v(t,x) = &\int_0^\infty \kappa_{OU}(t,x,y)f(y)\mathrm{d}y \\ &+ \int_0^\infty \kappa_{OU}(t,x,-y)\left[f(y) - 2he^{-hy}\int_0^y f(s)e^{hs}ds\right]\mathrm{d}y. \end{aligned} \tag{5.28}$$

Rearranging terms, we obtain the explicit solution

$$\begin{aligned} v(t,x) = &\int_0^\infty [\kappa_{OU}(t,x,y) + \kappa_{OU}(t,x,-y)]f(y)\mathrm{d}y \\ &- 2h\int_0^\infty \kappa_{OU}(t,x,-y)e^{-hy}\int_0^y f(s)e^{hs}\mathrm{d}s\mathrm{d}y. \end{aligned} \tag{5.29}$$

This leads us to the main result:

Theorem 5.5 *The unique solution* $v\colon (0,\infty)\times(0,\infty)\to\mathbb{K}$ *to the parabolic problem* (5.16)*,* (5.17) *and* (5.18) *with* decaying *Robin* BCs *is given by*

$$\begin{aligned} v(t,x) = &\int_0^\infty \left(\kappa_{OU}(t,x,y) + \kappa_{OU}(t,x,-y)\right) f(y)\mathrm{dy} \\ &- 2h\int_0^\infty \kappa_{OU}(t,x,-y)e^{-hy}\int_0^y f(s)e^{hs}\mathrm{dsdy}, \end{aligned} \tag{5.30}$$

and $\kappa_{OU}(t,x,y)$ *is the* OU *kernel* (3.24)*.*

Proof Existence follows from our explicit construction, and uniqueness follows from [LB06, Proposition 4.1.10]. ∎

The integral term $\int_0^y f(s)e^{hs}\mathrm{ds}$ in the solution reflects the non-local nature of the Robin condition, demonstrating how boundary effects propagate throughout the domain. It captures a memory effect, where the solution at any point depends on the entire history of boundary behavior.

5.4 CONCLUSION

The method developed in this chapter, exploiting operators H_Ω that commute with the OU evolution operator $\Box_{\mathcal{A}}$, provides an explicit solution to the parabolic problem (P_A) with time-dependent Robin BCs on $[0,\infty)$. This approach extends Bryan's technique for the heat equation to the OU setting, circumventing the non-commutativity of spatial derivatives with $\mathcal{A}$. While effective for $\boldsymbol{\mathcal{S}_1}$, the generalization to $\boldsymbol{\mathcal{S}_m}$ for $m>1$ remains open. Chapter 6 will explore a broader algebraic structure encompassing these commuting operators.

6

A LIE ALGEBRA FRAMEWORK FOR THE ORNSTEIN-UHLENBECK OPERATOR: THE T-EXTENDED ALGEBRA $\mathfrak{E}_T$

In every direction, the extension is endless;
the sensation of depth is overwhelming.
And the darkness is immortal.

— Carl Sagan [Sagkn]

6.1 GENERALIZATION TO HIGHER-ORDER OPERATORS

We extend the analysis of operators commuting with the Ornstein-Uhlenbeck (OU) evolution operator $\Box_{\mathcal{A}}$ to encompass linear operators of arbitrary order. This generalization not only yields a method for constructing novel solutions to the associated parabolic problem but also reveals a rich algebraic structure, denoted $\mathfrak{E}_T$, which we term the *T–extended algebra*.

First, let us recall some fundamental operators:

Definition 6.1 (Fundamental Operators) *For $f \in \mathcal{D}_\nu^n$, $\nu \in (0,1)$, $n \in \mathbb{N}$, define:*

$$\begin{aligned} \mathcal{I}f &:= f \\ Tf &:= \partial_t f \\ X_t f &:= e^t \partial f = e^t \partial_x f \\ X_t^* f &:= -e^{-t} \partial^* f = -e^{-t}(2xf - \partial_x f) \end{aligned}$$

where ∂ and ∂^* are the annihilation and creation operators.

Lemma 6.2 *The operators $\mathcal{I}$, T, X_t, and X_t^* commute with the OU evolution operator $\Box_{\mathcal{A}}$.*

Proof The commutation of $\mathcal{I}$ and T with $\Box_{\mathcal{A}}$ is immediate. For X_t and X_t^*, the proof follows directly from Proposition 5.3. The operators X_t, and X_t^* can be expressed as H_Ω for specific choices of the parameter ω_{ijk}, where $\omega_{ijk} = 1$ for a single index combination and 0 otherwise:

$$\begin{aligned} X_t &= H_\Omega \quad \text{for} \quad \omega_{010} = 1, \\ X_t^* &= H_\Omega \quad \text{for} \quad \omega_{100} = 1. \end{aligned}$$

Since Proposition 5.3 establishes that H_Ω commutes with $\Box_{\mathcal{A}}$ for any choice of ω_{ijk}, the result follows. ∎

We now extend our analysis to higher-order operators constructed from the fundamental operators introduced earlier.

Definition 6.3 *Let $n \in \mathbb{N}_0$ and $\Omega := \{\omega_{ijk} \in \mathbb{K} : i, j, k \in \mathbb{N}_0, i + j + k \leq n\}$, we define the higher-order operator H^n_Ω on $\mathcal{D}^n_\nu$, for $\nu \in (0, 1)$ as:*

$$H^n_\Omega := \sum_{i+j+k=0}^{n} \omega_{ijk} \cdot (X_t^*)^i X_t^j T^k \tag{6.1}$$

This definition allows us to construct operators of arbitrary order by taking linear combinations of products of our fundamental operators.

Proposition 6.4 *The operator $H^n_\Omega := \sum_{i+j+k=0}^{n} \omega_{ijk} \cdot (X_t^*)^i X_t^j T^k$ commutes with $\Box_\mathcal{A}$ for all $n \in \mathbb{N}_0$.*

Proof Recall that if an operator C commutes with $\Box_\mathcal{A}$, then all powers of C also commute with $\Box_\mathcal{A}$. From Lemma 6.2, we know that $\Box_\mathcal{A}$ commutes with $\mathcal{I}$,X_t^*, X_t, and T, and therefore with all powers of these operators.

For any $i, j, k \in \mathbb{N}_0$, we have:

$$\Box_\mathcal{A}(X_t^*)^i X_t^j T^k = (X_t^*)^i \Box_\mathcal{A} X_t^j T^k = (X_t^*)^i X_t^j \Box_\mathcal{A} T^k = (X_t^*)^i X_t^j T^k \Box_\mathcal{A}.$$

By linearity, this result extends to H^n_Ω. Therefore, $[\Box_\mathcal{A}, H^n_\Omega] = 0$, completing the proof. ∎

Remark 6.5 *This result extends Proposition 5.3 from first-order differential operators to arbitrary finite-order operators constructed from powers of X_t^*, X_t, and T, all commuting with $\Box_\mathcal{A}$.*

6.2 THE T–EXTENDED ALGEBRA $\mathfrak{E}_T$

The commutation properties of our fundamental operators naturally lead us to investigate the algebraic structure they generate.

Definition 6.6 (T–Extended Algebra $\mathfrak{E}_T$) *Let $\mathfrak{E}_T$ be the $\mathbb{K}$-vector space spanned by the operators $\mathcal{I}$, T, X_t, and X_t^*, where for $f \in \mathcal{D}^n_\nu$, $\nu \in (0, 1)$, $n \in \mathbb{N}$:*

$$\begin{aligned} \mathcal{I}f &= f, \\ Tf &= \partial_t f, \\ X_t f &= e^t \partial f, \\ X_t^* f &= e^{-t}(-\partial^*) f. \end{aligned}$$

We equip $\mathfrak{E}_T$ with the commutator bracket $[A, B] := AB - BA$ and refer to this space as the T-extended algebra.

Remark 6.7 *The algebra $\mathfrak{E}_T$ is termed an "extended" algebra because it contains a subalgebra isomorphic to the well-known Heisenberg algebra. Specifically, the generators $\mathcal{I}$, X_t^*, and X_t form a* Lie algebra *((see Appendix C.1) with the commutation relations:*

$$\begin{aligned} [X_t^*, X_t] &= 2\mathcal{I}, \\ [\mathcal{I}, X_t^*] &= [\mathcal{I}, X_t] = 0. \end{aligned}$$

These are precisely the defining relations of the Heisenberg algebra, with X_t^* and X_t playing the roles of the creation and annihilation operators, respectively. Note that $[X_t^*, X_t] = [-\partial^*, \partial] = 2\mathcal{I}$. The T-extended algebra $\mathfrak{E}_T$ can be seen as an extension of the Heisenberg algebra by including the time translation operator T.

Theorem 6.8 *The T*-extended algebra *$\mathfrak{E}_T$, equipped with the commutator bracket, forms a Lie algebra (see Appendix C.1). The non-trivial commutation relations are:*

$$
\begin{aligned}
[T, X_t] &= X_t \\
[T, X_t^*] &= -X_t^* \\
[X_t^*, X_t] &= 2\mathcal{I}
\end{aligned}
$$

Proof We first prove the commutation relations. Assume $f \in D_\nu^2$, for $\nu \in (0, 1)$.

$\underline{[T, X_t] = X_t}$:

$$
\begin{aligned}
[T, X_t]f &= (TX_t - X_tT)f \\
&= \partial_t(e^t\partial_x f) - e^t\partial_x(\partial_t f) \\
&= e^t\partial_x f + e^t\partial_t\partial_x f - e^t\partial_x\partial_t f \\
&= e^t\partial_x f = X_t f
\end{aligned}
$$

$\underline{[T, X_t^*] = -X_t^*}$:

$$
\begin{aligned}
[T, X_t^*]f &= (TX_t^* - X_t^*T)f \\
&= \partial_t(-e^{-t}(2xf - \partial_x f)) - (-e^{-t}(2x\partial_t f - \partial_x\partial_t f)) \\
&= e^{-t}(2xf - \partial_x f) + (-e^{-t}(2x\partial_t f - \partial_t\partial_x f)) \\
&- (-e^{-t}(2x\partial_t f - \partial_x\partial_t f)) \\
&= e^{-t}(2xf - \partial_x f) = -X_t^* f
\end{aligned}
$$

$\underline{[X_t^*, X_t] = 2\mathcal{I}}$:

$$
\begin{aligned}
[X_t^*, X_t]f &= (X_t^*X_t - X_tX_t^*)f \\
&= (-e^{-t}e^t(2x\partial_x f - \partial_x\partial_x f)) + e^te^{-t}\partial_x(2xf - \partial_x f) \\
&= -2x\partial_x f + \partial_x^2 f + 2f + 2x\partial_x f - \partial_x^2 f \\
&= 2f = 2\mathcal{I}f
\end{aligned}
$$

To show that $\mathfrak{E}_T$ is a Lie algebra, we must verify that the commutator bracket satisfies bilinearity, antisymmetry, and the Jacobi identity. The bilinearity and antisymmetry of the commutator bracket are immediate. For the Jacobi identity, we need only verify:

$$[T, [X_t, X_t^*]] + [X_t, [X_t^*, T]] + [X_t^*, [T, X_t]] = 0,$$

as all cases with repeated operators vanish identically. Indeed:

$$
\begin{aligned}
[T, [X_t, X_t^*]] + [X_t, [X_t^*, T]] + [X_t^*, [T, X_t]] &= [T, -2\mathcal{I}] + [X_t, X_t^*] + [X_t^*, X_t] \\
&= 0 + (-2\mathcal{I}) + 2\mathcal{I} = 0.
\end{aligned}
$$

Thus, $\mathfrak{E}_T$ satisfies all Lie algebra axioms. ∎

Remark 6.9 *This algebraic structure allows us to view Proposition 5.3 as a statement about the commutation of* $\Box_{\mathcal{A}}$ *with elements of the enveloping algebra of* $\mathfrak{E}_T$ *[Kir10, Definition 5.1].*

In our context, the universal enveloping algebra $U(\mathfrak{E}_T)$ *of* $\mathfrak{E}_T$ *is the associative algebra generated by the operators* $\mathcal{I}$*,* T*,* X_t*, and* X_t^**, subject to the commutation relations given in Theorem6.8. Elements of* $U(\mathfrak{E}_T)$ *are linear combinations of products of these operators, i.e., they are of the form:*

$$\sum_{i+j+k=0}^{n} c_{ijk}(X_t^*)^i X_t^j T^k$$

where $c_{ijk} \in \mathbb{K}$ *and* $n \in \mathbb{N}$*.*

The commutation relations of the T-extended algebra $\mathfrak{E}_T$ reveal a special element that commutes with all other elements of the algebra, known as a Casimir operator (see Definition C.2).

Corollary 6.10 *The* OU *evolution operator* $\Box_{\mathcal{A}}$ *is a Casimir operator of the* T*-extended algebra* $\mathfrak{E}_T$*.*

Proof The result follows directly from Lemma 6.2 and the definition of a Casimir operator (see Definition C.2). ∎

6.3 ISOMORPHISM TO THE OSCILLATOR ALGEBRA

We introduce the oscillator algebra $\mathfrak{O}^c$, a well-known algebraic structure in mathematical physics [KS97].

Definition 6.11 (Oscillator Algebra) *The oscillator algebra* $\mathfrak{O}^c$ *is generated by operators* $a, a^\dagger, N$ *and* $\mathcal{I}$ *with the commutation relations:*

$$[N, a^\dagger] = a^\dagger$$
$$[N, a] = -a$$
$$[a, a^\dagger] = \mathcal{I}$$
$$[\mathcal{I}, \cdot] = 0$$

The oscillator algebra plays a fundamental role in various areas of physics, including quantum mechanics, quantum optics, and the theory of special functions [KS97]. Remarkably, the T-extended algebra $\mathfrak{E}_T$ resembles $\mathfrak{O}^c$, as formalized by the following theorem.

Theorem 6.12 (Isomorphism to Oscillator Algebra) *The Lie algebra* $\mathfrak{E}_T$*, generated by the operators* $\{X_t, X_t^*, \mathcal{I}, T\}$ *with the commutation relations*

$$[T, X_t] = X_t$$
$$[T, X_t^*] = -X_t^*$$
$$[X_t, X_t^*] = 2\mathcal{I}$$
$$[\mathcal{I}, \cdot] = 0$$

is isomorphic to the oscillator algebra $\mathfrak{O}^c$*.*

Proof Define the normalized operators:

$$a^\dagger := \frac{X_t}{\sqrt{2}}$$
$$a = \frac{X_t^*}{\sqrt{2}}$$
$$N := T$$

Under this normalization, the commutation relations become:

$$[N, a^\dagger] = a^\dagger$$
$$[N, a] = -a$$
$$[a, a^\dagger] = \mathcal{I}$$
$$[\mathcal{I}, \cdot] = 0$$

These are precisely the defining relations of the oscillator algebra $\mathfrak{O}^c$ as given in Definition 6.11. ∎

The isomorphism between $\mathfrak{E}_T$ and $\mathfrak{O}^c$ provides a new perspective on the algebraic structure underlying the OU process. This connection suggests that techniques and results from the representation theory of $\mathfrak{O}^c$ could be utilized to gain new insights into the OU process and related parabolic problems.

6.4 SYNTHESIS OF ALGEBRAIC RESULTS

This chapter has elucidated the algebraic structure underlying the Ornstein-Uhlenbeck evolution operator $\square_{\mathcal{A}}$. The main results can be summarized as follows:

1. The commutation property $[\square_{\mathcal{A}}, H^n_\Omega] = 0$ holds for higher-order operators H^n_Ω.
2. The T-extended algebra $\mathfrak{E}_T = \text{span}\{\mathcal{I}, T, X_t, X_t^*\}$ forms a Lie algebra with $\square_{\mathcal{A}}$ as its Casimir operator.
3. There exists an isomorphism $\mathfrak{E}_T \cong \mathfrak{O}^c$, where $\mathfrak{O}^c$ is the oscillator algebra.

These findings establish an algebraic framework for analysing the OU operator and related parabolic problems.

7 CONCLUSION AND FUTURE DIRECTIONS

We shall not cease from exploration
The end of all our exploring
Will be to arrive where we started
And know the place for the first time.

— T.S. Eliot [Eli01]

7.1 SUMMARY OF RESULTS

This thesis has examined the Ornstein-Uhlenbeck (OU) operator on metric star graphs, extending previous results and unveiling new algebraic structures. The main contributions can be summarised as follows:

- We uncovered the key role of the even-odd extension method in translating results from the real line to metric star graphs. This approach, implicit in the work of Mugnolo and Rhandi [MR22], provides a systematic way to use known results for parabolic problems on $\mathbb{R}$ to establish the existence and uniqueness of solutions on metric star graphs.
- We extended the spectral analysis of the OU operator to include δ–Coupling Vertex Conditions (VCs) on metric star graphs. This generalization revealed a characterization of the eigenvalues in terms of a transcendental equation involving the δ–coupling strength.
- We developed a commutator-based approach for constructing new solutions to the parabolic problem. This method allows for generating solutions satisfying different boundary conditions by applying operators that commute with the OU evolution operator. We demonstrated its application to Robin boundary conditions on a semi-infinite interval, providing an explicit solution in terms of the OU kernel.
- We introduced the T-extended algebra $\mathfrak{E}_T$, unveiling the underlying algebraic structure of operators commuting with the OU evolution operator $\square_{\mathcal{A}}$. This algebra extends the well-known Heisenberg algebra, providing a richer framework for analyzing the OU process. Notably, we discovered that the OU evolution operator emerges as the Casimir operator of $\mathfrak{E}_T$.
- Finally, we established an isomorphism between $\mathfrak{E}_T$ and the oscillator algebra $\mathfrak{O}^c$, connecting our results to well-established structures in mathematical physics. This isomorphism facilitates applying techniques from the representation theory of $\mathfrak{O}^c$ to the study of the OU process.

7.2 FUTURE DIRECTIONS

Several potential avenues for future research emerge from this thesis:

1. **Extension to General Metric Graphs:** Investigate the applicability of our methods, particularly the even-odd extension technique and the commutator approach, to OU operators on more complex graph topologies.

2. **Further Exploration of δ-Coupling VCs:** Analyze solutions' spectral properties and long-time behavior under various δ-coupling strengths. This could involve a detailed study of how the coupling strength affects the eigenvalue distribution and the asymptotic behavior of solutions.

3. **Applications of the T-Extended Algebra:** Exploit the isomorphism with the oscillator algebra to derive new results about the OU process using techniques from representation theory. This could lead to novel insights into the structure of solutions and the behavior of the process.

4. **Stochastic Interpretation:** Develop a probabilistic interpretation of our results, particularly in the context of diffusion processes on metric graphs. This could involve studying the relationship between the algebraic structures we've uncovered and the sample path properties of the associated stochastic processes.

These directions offer the potential for deepening our understanding of the OU operator on metric graphs and its connections to broader areas of mathematics and physics.

Part I

APPENDIX

A

CONSERVATIVITY

Lemma A.1 *Let $\left(\boldsymbol{T_m}(t)\right)_{t\geq 0}$ be the semigroup on the metric star graph $\boldsymbol{\mathcal{S}_m}$ with the representation formula* (3.1) *and the integral kernel κ satisfying the identity* (3.23)*. Then, the semigroup $\left(\boldsymbol{T_m}(t)\right)_{t\geq 0}$ is conservative, i.e., $\boldsymbol{T_m}(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$.*

Proof

For $\boldsymbol{x}_i \in \boldsymbol{\mathcal{S}_m}$, $i \in I$, and $t > 0$, we have

$$
\begin{aligned}
\left(\boldsymbol{T_m}(t)\mathbb{1}\right)(\boldsymbol{x}_i) &= \int_{\boldsymbol{r}_i} \left(\kappa(t, |\boldsymbol{x}_i|, |\boldsymbol{\eta}_i|) - \kappa(t, |\boldsymbol{x}_i|, -|\boldsymbol{\eta}_i|)\right) \mathrm{d}\boldsymbol{\eta}_i \\
&\quad + \frac{2}{m} \sum_{j \in I} \int_{\boldsymbol{r}_j} \kappa(t, |\boldsymbol{x}_i|, -|\boldsymbol{\eta}_j|) \,\mathrm{d}\boldsymbol{\eta}_j \\
&\overset{(3.23)}{=} \int_{\boldsymbol{r}_i} \kappa(t, |\boldsymbol{x}_i|, |\boldsymbol{\eta}_i|) \,\mathrm{d}\boldsymbol{\eta}_i + \int_{\boldsymbol{r}_i} \kappa(t, |\boldsymbol{x}_i|, |\boldsymbol{\eta}_i|) \,\mathrm{d}\boldsymbol{\eta}_i - 1 \\
&\quad + \frac{2}{m} \sum_{j \in I} \left(1 - \int_{\boldsymbol{r}_i} \kappa(t, |\boldsymbol{x}_i|, |\boldsymbol{\eta}_i|) \,\mathrm{d}\boldsymbol{\eta}_i\right) \\
&= 2 \int_{\boldsymbol{r}_i} \kappa(t, |\boldsymbol{x}_i|, |\boldsymbol{\eta}_i|), \mathrm{d}\boldsymbol{\eta}_i + 1 - 2 \int_{\boldsymbol{r}_i} \kappa(t, |\boldsymbol{x}_i|, |\boldsymbol{\eta}_i|) \,\mathrm{d}\boldsymbol{\eta}_i = 1.
\end{aligned}
$$

Thus, since $\boldsymbol{T_m}(0) = \mathbb{1}$, $\boldsymbol{T_m}(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$. ∎

B

SINGULAR STURM-LIOUVILLE PROBLEMS: KEY DEFINITIONS AND THEOREMS

This appendix compiles the key definitions and theorems from singular Sturm-Liouville (SL) theory invoked throughout this thesis, drawing primarily from the contributions of [HK92] and [Wei03].

We introduce the differential operator τ, defined by the following expression:

$$\tau y := \frac{-(py')' + qy}{\gamma} \tag{B.1}$$

over an interval $(a, b), -\infty \leq a < b \leq \infty$, where p^{-1}, q, and $\gamma > 0$ are locally integrable within (a, b). This guarantees that there is only one solution to the equation $\tau y = f$, given suitable values of y and py' at every point $e \in (a, b)$.

Definition B.1 (Regular and Singular Endpoints) *An endpoint $x \in \{a, b\}$ is said to be* regular *if it is finite and the functions p^{-1}, q, and w are integrable on some neighbourhood of x within the interval $[a, b]$. Otherwise, x is called a* singular *point.*

Assume, following ([Wei03])

(i) q and ρ are piece-wise continuous real functions on (a, b), p is continuous and piece-wise continuous differentiable.

(ii) It is $\rho(x) > 0$ and $p(x) \neq 0$ for all $x \in (a, b)$.

Definition B.2 *For $g : (a, b) \to \mathbb{C}$ measurable and $z \in \mathbb{C}$, the function $f : (a, b) \to \mathbb{C}$ is called a solution of $(\tau - z)f = g$, if f and pf' are absolutely continuous (more precisely, they coincide almost everywhere with an absolutely continuous function) and*

$$-(pf')'(x) + (q(x) - z)\rho(x)f(x) = \rho(x)g(x)$$

holds almost everywhere for $x \in (a, b)$. Note that because of the (a') and (b') requirements, pf' is absolutely continuous if and only if f' is absolutely continuous.

Definition B.3 *For a SL equation $(\tau - z)u = 0$, the modified Wronskian of two solutions u_1 and u_2 is defined as*

$$W[u_1, u_2] := \det \begin{pmatrix} u_1(x) & u_2(x) \\ pu_1'(x) & pu_2'(x) \end{pmatrix} = u_1(x)pu_2'(x) - pu_1'(x)u_2(x), \tag{B.2}$$

where p is the coefficient function in the SL expression τ.

Definition B.4 *Let $f : (a, b) \to \mathbb{C}$ be measurable. we say that f lies to the left in $L^2_\gamma(a, b)$ if a $c \in (a, b)$ exists, such that $f|_{(a,b)} \in L^2_\gamma(a, c)$. Correspondingly, f lies to the right in $L^2_\gamma(a, b)$ if a $c \in (a, b)$ exists, such that $f|_{(a,b)} \in L^2_\gamma(c, b)$. Since we will use those definitions for solutions of $(\tau - z)u = 0$, that always applies to all or none $c \in (a, b)$.*

Theorem B.5 *Let τ be a* SL *expression in (a, b). If all solutions for a z_0 of $(\tau - z_0) = 0$ lie to the right in $L^2(a, b; r)$, then for all $z \in \mathbb{C}$ all solutions lie to the right in $L^2(a, b; r)$. The same applies to lying to the left.*

Theorem B.6 (The Weyl Alternative) *Consider a* SL *differential expression τ in (a, b). For each such τ, either holds*

(i) for each $z \in \mathbb{C}$ lie each solution u of $(\tau - z) = 0$ right in $L^2(a, b; r)$, or

(ii) for each $z \in \mathbb{C}$ exists at least a solution of $(\tau - z)u = 0$, that do not lye right in $L^2(a, b; r)$. In this case, exists for each $z \in \mathbb{C} \setminus \mathbb{R}$ exactly one solution of $(\tau - z) = 0$, that lies right in $L^2(a, b; r)$.

Following the original Weyl proof, the first case is called the limit circle, and the second is the limit point case. Those definitions are due to the methods used in Weyl proof.

C

ALGEBRAIC STRUCTURES

This appendix provides definitions and brief explanations of some algebraic structures used in the main text, namely Lie algebras and Casimir operators.

Definition C.1 (Lie Algebra) *A Lie algebra over* $\mathbb{K}$ *is a* $\mathbb{K}$*-vector space* $\mathfrak{g}$ *together with a bilinear operation* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, the Lie bracket, such that:*

1. ***Antisymmetry****:* $[u, v] = -[v, u]$ *for all* $u, v \in \mathfrak{g}$*.*
2. ***Jacobi identity****:* $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ *for all* $u, v, w \in \mathfrak{g}$*.*

Lie algebras play a pivotal role in diverse areas of mathematics and physics, often arising as the infinitesimal generators of Lie groups, which are smooth manifolds equipped with a group structure. Many important geometric and algebraic properties of Lie groups can be studied more easily by examining their associated Lie algebras.

Definition C.2 (Casimir Operator) *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathbb{K}$*. An element* $C \in \mathfrak{g}$ *is called a Casimir operator (or Casimir element) if* $[C, x] = 0$ *for all* $x \in \mathfrak{g}$*.*

Casimir operators are important in the study of representation theory of Lie algebras, as they provide a way to classify and construct representations. In physical applications, Casimir operators often correspond to conserved quantities or invariants of the system under consideration.

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