FernUniversität in Hagen
Faculty of Mathematics and Computer Science

Artificial Intelligence Group

An Investigation of Semantics Based on Cycles for AGM Revision in Propositional Logic

Master's Thesis

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submitted by
Florian Kauth

First examiner: Dr. Kai Sauerwald
Artificial Intelligence Group

Advisor: Dr. Kai Sauerwald
Artificial Intelligence Group

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Zusammenfassung

Für das Problem neues Wissen in eine bestehende Wissensmenge aufzunehmen, mit der es in Konflikt steht, gibt es verschiedene Ansätze. Einer der bekanntesten und einflussreichsten formalen Ansätze zur Wissensrevision von Wissensmengen ist die AGM-Theorie. In der AGM-Theorie wird neues Wissen bevorzugt in die Wissensmenge aufgenommen und Widersprüche werden so weit wie möglich aufgelöst. Die Axiomatisierung der AGM-Theorie besteht aus Postulaten, welche Anforderungen beschreiben, die ein Revisionsoperator erfüllen sollte. Diese Postulate werden zumeist in zwei Gruppen aufgeteilt, in die grundlegenden Postulate und die ergänzenden Postulate. Es ist bekannt, dass im Falle der Aussagenlogik, alle Postulate zusammen durch Totale Präordnungen charakterisiert werden. Außerdem ist bekannt, dass andere Gruppen von ergänzenden Postulaten zu anderen Ordnungen führen. Zyklische Ordnungen wurden bisher noch nicht in diesem Kontext betrachtet. Da ein Konzept von Minimalität in Kreisen nicht existiert, definieren wir ein Konzept von Nähe in zyklischen Ordnungen. Wir zeigen, dass es zyklische Ordnungen gibt, die Postulat (R8) von Katsuno und Mendelzon nicht erfüllen. Als Konsequenz zeigen wir, dass es eine Klasse von zyklischen Ordnungen gibt, die Postulat (R8) erfüllen und beweisen ein Repräsentationstheorem für eben diese Klasse. Außerdem untersuchen wir den Grund warum gewisse zyklische Ordnungen Postulat (R8) nicht erfüllen und definieren dabei Sperr-Mengen. Wir zeigen einige Eigenschaften dieser Sperr-Mengen im Kontext von zyklischen Ordnungen und schlagen Postulate vor, die diese Eigenschaften ausdrücken sollen.

Abstract

The field of belief change studies how the beliefs of an agent can be rationally changed in the light of new information. An influential formal framework for belief change is AGM theory. One of the primary operations considered by AGM theory is revision, which is the kind of belief change where a new belief is incorporated into the initial beliefs such that new information is prioritized and inconsistencies are solved whenever it is possible. The axiomatization of AGM revision consists of postulates which are typically split into two groups, the basic postulates and the supplementary postulates. A well known result is that, in the case of propositional logic, the full set of AGM postulates for revision is characterized by total preorders over the interpretations. It is also known that replacing the supplementary postulates with other postulates leads to characterizations by different types of orderings than total preorders. Cyclic orders were not studied in the context of belief change before. We address the problem that the concept of minimality is not defined for cycles. Instead we define a concept of closeness in cyclic orders. We show that there are cyclic orders which do not satisfy the Katsuno and Mendelzon postulate (R8). As a result we establish a class of cyclic orders that do satisfy postulate (R8) and

proof a representation theorem for this class. We investigate the reason why some cyclic orders do not satisfy postulate (R8) and define blocking sets in the process. We show some properties of blocking sets in cyclic orders and introduce potential candidates for postulates to express these properties.

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1 Introduction

In knowledge representation, the field of belief change studies how the beliefs of an agent can change rationally (in the light of new information) [FH18]. A prominent theory is AGM theory, which is centered on the idea that belief changes should be minimal in the sense that agents should keep as many as possible of their initial beliefs. One of the primary operations considered by AGM theory is revision, which is the kind of belief change where a new belief is incorporated into the initial beliefs such that the new information is prioritized and inconsistencies are solved whenever that is possible. The axiomatization of AGM revision used today consists of postulates which are typically split into two groups, the basic postulates and the supplementary postulates. Some authors argue that the basic postulates already capture minimal change, and the supplementary postulates represent certain organizational principles regarding the interrelation of beliefs. Formally, Katsuno and Mendelzon [KM91] showed that AGM revision (with all postulates) is characterized by total preorders over the interpretations (when considering propositional logic). It is also known that replacing the supplementary postulates with other postulates leads to characterizations by different types of orderings than total preorders, e.g., semi-orders [PW14]. Today, for many types of orderings, a corresponding set of postulates C is unknown.

One type of orderings, for which a corresponding set of postulates is unknown, is the type of cyclic orders. The goal of this thesis is to formulate a representation theorem for a set of postulates and cyclic orders similar to the representation theorem by Katsuno and Mendelzon for total preorders (see [KM91]). It is well known that a cyclic order cannot be defined binary [Nov82]. Therefore we use the definition of cyclic orders as ternary relations in this thesis. Due to the close relation of partial orders and (partial) cyclic orders we first examine the state of the postulates for partial orders and attempt to adopt these postulates to (partial) cyclic orders. In order to do this we need to first adopt the other parts of the representation theorem to cyclic orders. Alongside the set of postulates these parts consist of a notion of minimality and a class of functions called faithful assignments. Because minimality is not a concept that can be easily transferred to cycles we define a concept of *closeness* for cyclic orders. However this concept of closeness in cyclic orders cannot entirely take the place of the concept of closeness in binary orders used by Katsuno and Mendelzon. Therefore we use an easy fix to ensure postulate (R2). The definition of faithful assignments defines the set of orders in the context of the representation theorem. Some authors argue that the definition of faithful assignments for partial orders, by Katsuno and Mendelzon, is too restrictive [BLP05]. Benferhat, Lagrue and Papini present a slightly different set of postulates and a different definition of faithful assignments for partial orders. Because their changes include one of the basic postulates we choose the set of postulates defined by Katsuno and Mendelzon. It is noteworthy that both argue for postulate (R8). For our goal of the definition of a representation theorem for cyclic orders we start with a very lenient definition of faithful cyclic assignments. In fact we allow all cyclic orders. Together with our notion of closeness in cyclic orders we find a counterexample for (R8). Consequently we define a more restrictive version of faithful assignments which we call *strong faithful cyclic assignments*. After we show the representation theorem for strong faithful cyclic orders we investigate the class of cyclic orders that do not satisfy (R8). We isolate a property which we give the name *blocking sets*. A blocking set for some possible world ω is a set of possible worlds such that ω is not 'close' if it is together with its blocking set, however it is 'close' if it is together with each real subset of its blocking set. We define a few postulates to ensure that blocking sets obey some principles and proof that they are applicable.

2 Preliminaries

2.1 Formal Preliminaries

Throughout this thesis we work with a signature Σ, which is a finite set whose elements we call propositional variables. The language of propositional logic $\mathcal{L}$ over Σ is the smallest set that contains Σ, the usual truth connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and for all $\alpha, \beta \in \mathcal{L}$:

1. $\neg\, \alpha \in \mathcal{L}$,
2. $\alpha \vee \beta \in \mathcal{L}$,
3. $\alpha \wedge \beta \in \mathcal{L}$.

$\perp$ denotes an arbitrary contradiction and $\top$ an arbitrary tautology.

An *interpretation* of $\mathcal{L}$, is a function $\mathrm{I} : \Sigma \to \{0, 1\}$ (alternatively $\{\mathrm{T}, \mathrm{F}\}$ or $\{true,\ false\}$ instead of $\{0, 1\}$). The valuation of a propositional formula α by an interpretation I, written $[[\alpha]]_{\mathrm{I}}$, is defined by:

1. If $\alpha \in \Sigma$ then $[[\alpha]]_{\mathrm{I}} = \mathrm{I}(\alpha)$,
2. If $\alpha = \neg\beta$, then $[[\alpha]]_{\mathrm{I}} = 1$ if $[[\beta]]_{\mathrm{I}} = 0$; otherwise $[[\alpha]]_{\mathrm{I}} = 0$,
3. If $\alpha = \beta \vee \gamma$, then $[[\alpha]]_{\mathrm{I}} = 1$ if $[[\beta]]_{\mathrm{I}} = 1$ or $[[\gamma]]_{\mathrm{I}} = 1$; otherwise $[[\alpha]]_{\mathrm{I}} = 0$,
4. If $\alpha = \beta \wedge \gamma$, then $[[\alpha]]_{\mathrm{I}} = 1$ if $[[\beta]]_{\mathrm{I}} = 1$ and $[[\gamma]]_{\mathrm{I}} = 1$; otherwise $[[\alpha]]_{\mathrm{I}} = 0$.

We denote by Ω the set of interpretations of $\mathcal{L}$. When considering propositional logic, an interpretation $\omega \in \Omega$ is often called a possible world. We sometimes denote a possible world by a tuple representing each propositional variables value, e.g. if $\Sigma = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ then $\langle 0, 1, 1\rangle$ is the interpretation which maps a,b,c to 0,1,1. For a propositional formula $\psi \in \mathcal{L}$ a possible world $\omega \in \Omega$ such that $[[\psi]]_{\omega} = 1$ is called a model of ψ and we write $\omega \models \psi$. We denote by $\mathrm{Mod}(\psi) \subseteq \Omega$ the set of all the models of ψ (i.e. $\mathrm{Mod}(\psi) = \{\omega |\ \omega \in \Omega : \omega \models \psi\}$). A propositional formula $\psi \in \mathcal{L}$ is called satisfiable if $\mathrm{Mod}(\psi) \neq \emptyset$ and unsatisfiable if $\mathrm{Mod}(\psi) = \emptyset$

For a propositional formula $\psi \in \mathcal{L}$ we denote by $Cn(\psi)$ the set of all logical consequences of ψ, i.e. $Cn(\psi) = \{\mu \in \mathcal{L} \mid \psi \models \mu\}$. For $\psi, \phi \in \mathcal{L}$ we denote $\psi \equiv \phi$ iff $\mathrm{Mod}(\psi) = \mathrm{Mod}(\phi)$. A propositional formula $\psi \in \mathcal{L}$ implies a propositional formula $\mu \in \mathcal{L}$ iff $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\mu)$. Furthermore for $\mu \in \mathcal{L}$ and $\phi \in \mathcal{L}$ we know that $\mathrm{Mod}(\mu \wedge \phi) = \mathrm{Mod}(\mu) \cap \mathrm{Mod}(\phi)$ and $\mathrm{Mod}(\mu \vee \phi) = \mathrm{Mod}(\mu) \cup \mathrm{Mod}(\phi)$ hold. For $\Omega' \subseteq \Omega$ we denote by $\mathrm{form}(\Omega')$ a formula whose set of models is equal to Ω'. For a set of propositional formulas $\mathcal{A} \subseteq \mathcal{L}$ we denote by $\bigvee \mathcal{A}$ the disjunction of all formulas of $\mathcal{A}$ and by $\bigwedge \mathcal{A}$ the conjunction of all formulas of $\mathcal{A}$.

2.2 AGM-Postulates and minimal change

The paper "On the Logic of Theory Change: Partial Meet Contraction and Revision Functions"[AGM85] by Carlos Alchourrón, Peter Gärdenfors and David Makinson is one of the most influential works on belief change. In the paper the authors investigated what properties a belief change operator should have to be desirable. One of the primary belief change operations considered in the paper is belief revision. As a result the authors defined a set of postulates a belief revision function should satisfy. Their formal framework is called AGM theory (after their initials) and the postulates are called AGM postulates accordingly.

In another important paper "Propositional knowledge base revision and minimal change"[KM91] by Hirofumi Katsuno and Alberto Mendelzon show a representation theorem for the AGM postulates. Because Katsuno and Mendelzon use knowledge bases instead of theories (deductively closed sets of sentences) they reformulated the AGM postulates for revision. As the goal of this thesis is a representation theorem for cyclic orders, similar to the one proposed by Katsuno and Mendelzon, we will use their version of the AGM postulates as well.

A knowledge base ψ is a sentence in $\mathcal{L}$ and for a sentence $\mu \in \mathcal{L}$, $\psi \circ \mu$ denotes the revision of ψ by μ. The result of the revision of $\psi \in \mathcal{L}$ with $\mu \in \mathcal{L}$ is a new knowledge base i.e. $(\psi \circ \mu) \in \mathcal{L}$. Instead of the first six AGM postulates Katsuno and Mendelzon propose the following four:

(R1) $\psi \circ \mu$ implies μ.
(R2) If $\psi \wedge \mu$ is satisfiable, then $\psi \circ \mu \equiv \psi \wedge \mu$.
(R3) If μ is satisfiable, then $\psi \circ \mu$ is also satisfiable.
(R4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 \circ \mu_1 \equiv \psi_2 \circ \mu_2$.

The intuitive meaning of postulate (R1) is that the new information must be maintained by the new knowledge base. Because (R3) demands that the new knowledge base must be satisfiable if the new information is satisfiable, it follows that the new information has priority and any beliefs in the old knowledge base, that contradict μ must be thrown out. Postulate (R3) does also prevent a revision from introducing unwarranted inconsistency [KM91]. If the knowledge base and the new information do not contradict each other, then (R2) ensures that the revision performs the obvious monotonic update. Postulate (R4) demands that the revision is not impacted by the way the knowledge is expressed in. This is often called "Dalal's Principle of Irrelevance of Syntax" [KM91] and not always agreed upon.

The next two postulates are the supplementary postulates that characterize the notion of minimal change in the AGM framework [KM91]:

(R5) $(\psi \circ \mu) \wedge \phi$ implies $\psi \circ (\mu \wedge \phi)$.
(R6) If $(\psi \circ \mu) \wedge \phi$ is satisfiable, then $\psi \circ (\mu \wedge \phi)$ implies $(\psi \circ \mu) \wedge \phi$.

We repeat the explanation of (R5) and (R6) by Katsuno and Mendelzon [KM91]: We assume that there is some metric for measuring the "distance" between the models of the knowledge base ψ and all possible worlds $\omega \in \Omega$. The revision operator should follow the principle of minimal change, which means that the models of $\psi \circ \mu$ should be the models of μ with that are closest to the models of ψ with respect to the distance metric.

According to Katsuno and Mendelzon rule (R5) says that the notion of closeness is well-behaved in the sense that any possible world ω which is closest to $\mathrm{Mod}(\psi)$ in a set, here $\mathrm{Mod}(\mu)$, and ω also belongs to a smaller set, $\mathrm{Mod}(\mu \wedge \phi)$, then ω must also be closest to $\mathrm{Mod}(\psi)$ within the smaller set $\mathrm{Mod}(\psi \wedge \mu)$.

Rule (R6) guarantees that no possible world ω_1 may be closer to ψ than ω_2 within a certain set, while ω_2 is closer than ω_1 within some other set.

The representation theorem by Katsuno and Mendelzon for the AGM postulates concerns orders on the set of possible worlds. Therefore we need to define orders first. We start with the definition of properties of binary relations:

Definition 1. *Let G be a set, a binary relation $\mathcal{R} \subseteq \mathrm{G} \times \mathrm{G}$ on G is called*

- transitive *if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ implies $(x, z) \in \mathcal{R}$,*
- reflexive *if for all $x \in$G: $(x, x) \in \mathcal{R}$,*
- irreflexive *if for all $x \in$G: $(x, x) \notin \mathcal{R}$,*
- symmetric *if $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$,*
- asymmetric *if $(x, y) \in \mathcal{R}$ implies $(y, x) \notin \mathcal{R}$,*
- antisymmetric *if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ then $x = y$,*
- connected *if for all $x, y \in$G: $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$.*

With these properties we can define the orders of interest:

Definition 2. *Let $\preceq$ be a binary relation on a set G, then $\preceq$ is called*

1. *a* preorder, *if it is reflexive and transitive,*
2. *a* total preorder, *if it is reflexive, transitive and connected,*

3. *a* partial order*, if it is reflexive, transitive and antisymmetric,*

4. *a* strict partial order*, if it is irreflexive, transitive and asymmetric.*

For $\mathrm{a}, \mathrm{b} \in \mathrm{G}$ we will use $\mathrm{a} \leqslant \mathrm{b}$ instead of $(\mathrm{a}, \mathrm{b}) \in \mathcal{R}$ if $\mathcal{R}$ is one of the orders defined above. Furthermore we define < as $\mathrm{a} < \mathrm{b}$ if and only if $\mathrm{a} \leqslant \mathrm{b}$ and $\mathrm{b} \nleqslant \mathrm{a}$.

A known way to graphically represent partial orders are directed graphs:

Definition 3. *A directed Graph* G *is a tuple* $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ *where* V *and* E *is a binary relation on V. An element of* V *is called a vertex or a node, while an element of* E*, i.e.,* $(\mathrm{v}_1, \mathrm{v}_2) \in \mathrm{E},\ \mathrm{v}_1, \mathrm{v}_2 \in \mathrm{V}$ *is called a directed edge or arrow.*

As previously stated, the AGM postulates demand that revision operators should adhere to the principle of minimal change. Katsuno and Mendelzon use the term 'closest' in relation to possible worlds. In order to characterize minimal change on the set of possible worlds with respect to some total pre-order, a metric of closeness is needed. For $\Omega' \subseteq \Omega$ Katsuno and Mendelzon define ω to be minimal in Ω' with respect to $\leqslant_\psi$ if $\omega \in \Omega'$ and there is no $\omega' \in \Omega'$ such that $\omega' <_\psi \omega$ and note:

$$\mathrm{Min}(\Omega', \leqslant_\psi) = \{\omega \mid \omega \text{ is minimal in } \Omega' \text{ with respect to } \leqslant_\psi\}$$

In order to characterize (R1)-(R6) by total preorders on possible worlds, Katsuno and Mendelzon define the term *faithful assignment* [KM91]:

Definition 4. *A* faithful assignment *is a function, that assigns to each propositional formula* $\psi \in \mathcal{L}$ *a total preorder* $\leqslant_\psi$ *and satisfies the following conditions:*

1. *If* $\omega, \omega' \in Mod(\psi)$*, then* $\omega \not<_\psi \omega'$

2. *If* $\omega \in Mod(\psi)$ *and* $\omega' \notin Mod(\psi)$*, then* $\omega <_\psi \omega'$

3. *If* $\psi \equiv \phi$ *then* $\leqslant_\psi = \leqslant_\phi$

That means a model of ψ cannot be strictly less than any other model of ψ and must be strictly less than any possible world, that is not a model of ψ [KM91]. When Katsuno and Mendelzon state: "If we regard $\leqslant_\psi$ as a measure representing the closeness between $\mathrm{Mod}(\psi)$ and an interpretation, i.e., $\mathrm{I}^{'} \leqslant_\psi \mathrm{I}$ means that $\mathrm{I}^{'}$ is closer to $\mathrm{Mod}(\psi)$ than I, then $\mathrm{Min}(\mathcal{M}, \leqslant_\psi)$ can be seen as the set of all the closest interpretations in $\mathcal{M}$ to $\mathrm{Mod}(\psi)$"[KM91], they probably mean $\mathrm{I}' <_\psi \mathrm{I}$.

The connection between AGM revision, faithful assignments and total preorders is given by the following theorem.

Theorem 1. *Revision operator* $\circ$ *satisfies conditions (R1)-(R6) if and only if there exists a faithful assignment that maps each KB* ψ *to a total preorder* $\leqslant_\psi$ *such that* $Mod(\psi \circ \mu) = Min(Mod(\mu), \leqslant_\psi)$ *[KM91].*

This theorem gives a characterization of AGM revision in model theory (when considering propositional logic). The goal of this thesis is the formulation of a representation theorem for cyclic orders similar to this theorem.

2.3 Belief Change on other orderings

It is important to note that the faithful assignment, used in the representation theorem, maps knowledge bases to *total* pre-orders [KM91]. The connectedness, of a total pre-order means, that all possible worlds are comparable. It is contentious whether this is a desirable property. Some argue for orders where two possible worlds can be incomparable (see [KM91]). However the totality or connectedness of a total pre-order is needed to guarantee (R6) [KM91, PW14, BLP05].

In the same paper, Katsuno and Mendelzon themselves have shown alternative supplementary postulates that characterize revision by partial pre-orders. They have also shown that when we consider partial pre-orders and partial orders, the difference between partial orders and partial pre-orders disappears [KM91]. In both cases the following (R7) and (R8) replace (R6):

(R7) If $\psi \circ \mu_1$ implies μ_2 and $\psi \circ \mu_2$ implies μ_1, then $\psi \circ \mu_1$ is equivalent to $\psi \circ \mu_2$.
(R8) $(\psi \circ \mu_1) \wedge (\psi \circ \mu_2)$ implies $\psi \circ (\mu_1 \vee \mu_2)$.

It is implied by Katsuno and Mendelzon, that the minimality as well as the faithful assignment are defined the same way as for total pre-orders. And with these two components they define a representation theorem for partial (pre-)orders:

Theorem 2. *Revision operator $\circ$ satisfies conditions (R1)-(R5), (R7) and (R8) if and only if there exists a faithful assignment that maps each KB ψ to a partial (pre-) order $\leqslant_\psi$ such that $Mod(\psi \circ \mu) = Min(Mod(\mu), \leqslant_\psi)$.*

Some authors argue that this representation theorem is "not satisfactory since only one class of partial pre-orders can be revised" [BLP05]. Salem Benferhat and Sylvain Lagrue give the following example of a very simple partial pre-order:

Example 1. *Let $\Omega = \{\omega_0, \omega_1, \omega_2, \omega_3\}$,*

$$\omega_0 <_\psi \omega_1 <_\psi \omega_2$$

and ω_3 is incomparable with ω_0, ω_1, ω_2. Then ω_1 and ω_3 are the models of ψ because no other possible world is strictly preferred to ω_0 and ω_3. However Katsuno and Mendelzon demand that $\omega_\psi <_\psi \omega$ if $\omega_\psi \in \mathrm{Mod}(\psi)$ and $\omega \notin \mathrm{Mod}(\psi)$. Benferhat and Lagrue state that this "$\leqslant_\psi$ is not a faithful assignment in the sense of Katsuno and Mendelzon". We instead say that this partial pre-order is not a faithful partial pre-order *in the sense of Katsuno and Mendelzon, to distinguish the order from a faithful assignment, which is (as definded by Katsuno and Mendelzon) a function that assigns to each propositional formula $\psi \in \mathcal{L}$ a (total/partial) pre-order.*

Benferhat, Lagrue and Papini argue that it is desirable to allow partial orders like the one defined in the example. For this purpose they formulate a slightly different set of postulates:

Definition 5. *Knowledge base $\psi \in \mathcal{L}$ new information μ*
(P1) $\psi \circ \mu$ implies μ,
(P2) $(\psi \circ \top) \equiv \psi$,
(P3) if μ is satisfiable, then $\psi \circ \mu$ is satisfiable,
(P4) if $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$ then $\psi_1 \circ \mu_1 \equiv \psi_2 \circ \mu_2$,
(P5) $(\psi \circ \mu) \wedge \phi$ implies $\psi \circ (\mu \wedge \phi)$,
(P6) if $\psi \circ \mu_1$ implies μ_2 and $\psi \circ \mu_2$ implies μ_1 then $(\psi \circ \mu_1) \equiv (\psi \circ \mu_2)$,
(P7) $(\psi \circ \mu_1) \wedge (\psi \circ \mu_2)$ implies $(\psi \circ (\mu_1 \vee \mu_2))$.

[BLP05]

These postulates are not the exact postulates proposed by Benferhat, Lagrue and Papini because they use epistemic states. Since they also focus on iteration, instead of (R4) they use the version by Adnan Darwiche and Judea Pearl, i.e., if $\Psi_1 = \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Psi_1 \circ \mu_1 \equiv \Psi_2 \circ \mu_2$ [DP94]. Because we do not focus on iteration, we do not need epistemic states and therefore simply put the version of Katsuno and Mendelzon instead. We see that the only other change concerns (R2), because when we consider the partial order in example 1 and a revision with $\mu = \mathrm{form}(\omega_1, \omega_3)$ the result would be $\psi \circ \mu \equiv \mathrm{form}(\omega_1, \omega_3)$ even though (R2) demands $\psi \circ \mu \equiv \mathrm{form}(\omega_3)$.

Benferhat and Lagrue therefore argue against the basic postulates which is the main reason why we continue with the postulates defined by Katsuno and Mendelzon.

The motivation to consider partial pre-orders instead of total pre-orders, lies in the possibility that two possible worlds are not comparable. A more recent paper by Pavlos Peppas and Mary-Anne Williams argues that in the AGM framework the "indifference of comparative plausibility is transitive"[PW14]. Because others have argued that semiorders are "an adequate model for human preference"[PW14] they have formulated a set of postulates and a representation theorem for semiorders and said set of postulates. It is noteworthy that Peppas and Williams incorporate postulates (R1) to (R5) and (R8) [1]. In contrast to Katsuno and Mendelzon's set of postulates for partial (pre-)orders they omit (R7) which is understandable given the two alternatives for (R7):

(R6w) If $\psi \circ \mu$ implies ϕ, then $\psi \circ (\mu \wedge \phi)$ implies $(\psi \circ \mu) \wedge \phi$,
(Rt) If $\psi \circ (\mu_1 \vee \mu_2) \equiv \mu_1$ and $\psi \circ (\mu_2 \vee \mu_3) \equiv \mu_2$, then $\psi \circ (\mu_1 \vee \mu_3) \equiv \mu_1$

and the following lemma by Katsuno and Mendelzon:

Lemma 3. *Assume that a revision operator $\circ$ satisfies (R1) to (R5). Then the following three conditions are equivalent.*

1. *The revision operator $\circ$ satisfies (R7).*
2. *The revision operator $\circ$ satisfies (R6w).*

[1] They give the credit for (R8) to Benferhat, Lagrue and Papini citing [BLP05]

3. *The revision operator $\circ$ satisfies (Rt).*

[KM91]

As we see "(Rt) intuitively guarantees transitivity"[KM91] which is not needed for semiorders. Instead Peppas and Williams define two additional postulates.

While the authors of these papers have presented arguments for the use of their respective orders, our motivation for finding a representation theorem for cyclic orders is not caused by any argument for the use of cyclic orders in belief-change. Instead we investigate belief-change defined by cyclic orders, to lay the foundation in case someone finds arguments for the use of cyclic orders.

2.4 Cyclic Orders

"It is well known that it is impossible to define an orientation of a circle by means of binary relation, but it is sufficient to use a ternary relation"[Nov82]. There are other ways to express such cyclic orientations. However since we want to formulate a representation theorem for cyclic orders, that is similar to the ones defined for binary orders, we choose the definition that is closely related.

Definition 6. *Let G be a set. A ternary relation T on the set G is any subset of $G \times G \times G$. We call T:*

1. ***cyclic*** *if* $(x, y, z) \in T$ *implies* $(y, z, x) \in T$.
2. ***asymmetric*** *if* $(x, y, z) \in T$ *implies* $(x, z, y) \notin T$.
3. ***transitive*** *if* $(x, y, z) \in T$ *and* $(x, z, u) \in T$ *then* $(x, y, u) \in T$ *and* $(y, z, u) \in T$.
4. ***connected*** *if for all* $x, y, z \in G$ *with* $x \neq y \neq z \neq x$ *exists a permutation σ such that* $\sigma(x, y, z) \in T$.

It is important to note that the above definitions are not complete, but it would be excessive to consider all permutations in the definitions. The cyclic condition does cause permutations in particular. The following theorem shows the impact cyclicity has on transitivity:

Theorem 4. *Let G be a set, T a cyclic ternary relation on G. T is transitive if and only if one of the following equivalent conditions holds:*

1. $(x, y, z) \in T, (x, u, y) \in T \Rightarrow (x, u, z) \in T$,
2. $(x, y, z) \in T, (x, u, y) \in T \Rightarrow (u, y, z) \in T$,
3. $(x, y, z) \in T, (y, u, z) \in T \Rightarrow (x, y, u) \in T$,
4. $(x, y, z) \in T, (y, u, z) \in T \Rightarrow (x, u, z) \in T$

[Nov82]

As the names, asymmetry, transitivity and connectedness suggest, there is a strong connection between binary relations and ternary relations. In particular, for every point of a ternary relation, we can define a binary relation:

Definition 7. *Let G be a set, let T be a ternary relation on G and let* $x_0 \in G$*. We denote by* ϱ_{T,x_0} *the binary relation on G defined as follows:*

$$(x, y) \in \varrho_{T,x_0} \Leftrightarrow (x_0, x, y) \in T.$$

[Nov82]

We see the strong connection between binary properties and ternary properties:

Theorem 5. *Let G be a set, let T be a ternary relation on G. Then:*

1. ϱ_{T,x_0} *is a transitive binary relation on G for each* $x_0 \in G$ *if and only if the ternary relation on T is transitive.*

2. *If T is cyclic then* ϱ_{T,x_0} *is an asymmetric binary relation on G for each* $x_0 \in G$ *if and only if T is asymmetric.*

[Nov82]

In other words a ternary relation T on G is (ternary-)transitive if and only if for all $x_0 \in G$ the binary relations ϱ_{T,x_0} on G are (binary-)transitive. A ternary relation T on G is (ternary-)asymmetric if and only if for all $x_0 \in G$ the binary relations ϱ_{T,x_0} on G are (binary-)asymmetric.

As the name implies, a cyclic order is therefore a ternary transitive and asymmetric relation that is also cyclic:

Definition 8. *Let G be a set,* $\mathcal{C}$ *a ternary relation on G. We call* $\mathcal{C}$ *a* partial cyclic order *on G if it is asymmetric, transitive and cyclic. We call* $\mathcal{C}$ *a* complete cyclic order *on the set G if it is a partial cyclic order on G and complete.*

As a consequence of theorem 7:

Theorem 6. *Let* $(G, \mathcal{C})$ *be a cyclically ordered set and* $x_0 \in G$*. For any* $x, y \in G$ *put* $x <_{\mathcal{C},x_0} y \Leftrightarrow (x_0, x, y) \in \mathcal{C}$ or $x_0 = x \neq y$*. Then* $<_{\mathcal{C},x_0}$ *is a strict partial order on G with the least element* x_0*.*

[Nov82]

While we have a way of receiving binary orders from a cyclic order for every element, there is also a way of receiving a cyclic order via a binary (strict) partial order:

Theorem 7. *Let G be a set, let < be an strict partial order on G. Then the ternary relation* $\mathcal{C}_<$ *on G defined by* $(x, y, z) \in \mathcal{C}_< \Leftrightarrow x < y < z$ or $y < z < x$ or $z < x < y$ *is a cyclic order on G.*

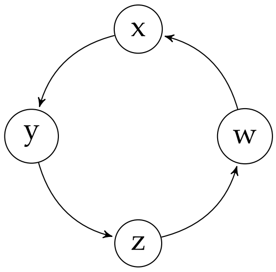

Figure 1: Cyclic order in example 3

[Nov82]

While binary partial orders can easily be depicted by directed graphs, it is generally not so easy to do the same for partial cyclic orders. A complete cyclic order can be depicted by a directed graph that is ordered in a circle as the following example shows:

Example 2. *Let* $\mathcal{C}_1 =$

$$\begin{gathered}\{(x,y,z),(x,y,w),(x,z,w),\\ (y,z,w),(y,z,x),(y,w,x),\\ (z,w,x),(z,w,y),(z,x,y),\\ (w,x,y),(w,x,z),(w,y,z)\}\end{gathered}$$

then we can draw a directed graph $G_1 = (V_1, E_1)$*, where* $V_1 = \{x, y, z, w\}$ *and* $E_1 = \{(x,y),(y,z),(z,w),(w,x)\}$*:*

At this point it is worth noting that the cyclic order above can be received from the set $\mathcal{M}_1 = \{(x,y,z),(x,z,w)\}$. The smallest asymmetric, transitive and cyclic set that includes $\mathcal{M}_1$ is $\mathcal{C}_1$. We omit the proof of this, since it is simply the computation of the transitive and cyclic cases. Because the cyclic permutations (and sometimes other tuples) mostly do not contain any additional knowledge we will often write something like $\mathcal{C}_1 = \{(x,y,z),(x,y,w),(x,z,w)\}$ is a cyclic order. With that we mean, that these tuples are sufficient to identify the smallest cyclic order that contains them.

There are proposals for oriented graphs for cyclic orders like the following. Let $\mathcal{C}$ be a cyclic order on a set X, a directed graph G=(V,E) defined by:

$$V = X,\ E = \{(x,y)|\ \exists\, z \in X : (x,y,z) \in \mathcal{C}\} \text{ (see [Qui89]).}$$

This proposed directed graph is not equal to the graph $G_1 = (V_1, E_1)$ pictured in example 3 because $(x,z,w) \in \mathcal{C}_1$ implies $(x,z) \in E$. Instead we will be somewhat ambiguous in our representation of partial cyclic orders, as the next example illustrates:

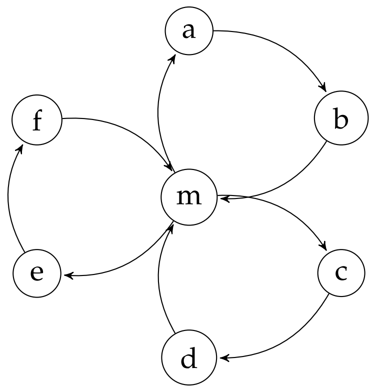

Figure 2: Cyclic order in example 3

Example 3. *Let* $\mathcal{C}_2 = \{(\mathrm{m}, \mathrm{a}, \mathrm{b}), (\mathrm{m}, \mathrm{c}, \mathrm{d}), (\mathrm{m}, \mathrm{e}, \mathrm{f})\}$*. With the following graph* G_2 *we give a representation that tries to capture a somewhat intuitive understanding:*

We want to emphasize that our graphical representations of partial cyclic orders do not follow the intuitive interpretation. Someone who is familiar with binary orders and their representation by directed graphs, would interpret that e and c are comparable because $(\mathrm{f}, \mathrm{m}) \in \mathrm{E}_2$ and $(\mathrm{m}, \mathrm{c}) \in \mathrm{E}_2$. Instead we want to interpret it as f and c do not lie on the same circle and therefore are not comparable. In general we want to note that directed graphs are not the best suited to represent partial cyclic orders.

3 Cyclic order revision

The goal of this thesis is to formulate a representation theorem for cyclic orders similar to theorem 1. We can distinguish three essential components in the theorem by Katsuno and Mendelzon:

1. The postulates (R1) to (R6) consisting of the basic postulates (R1) to (R4) and the supplemental postulates (R5) and (R6),
2. A class of functions called *faithful assignments*,
3. A concept of *minimality* or *closeness*.

Because of theorem 7 the most obvious candidates for postulates characterising cyclic orders, are the postulates used for partial orders (R1) to (R5), (R7) and (R8). However for the other two components we need to find fitting definitions. Therefore our approach is to first define a sensible notion of closeness in cyclic orders, then a definition of cyclic faithful assignments, that allows as many cyclic orders as possible and finally we need to either proof a representation theorem, or find a counterexample.

3.1 Minimality and cyclic orders

While a (partial) order can have a set of minimal elements (in our finite case every time) and cyclic orders can be constructed from such orders; the resulting cyclic orders do not indicate whether an element is minimal in the original binary order. For our goal of representing belief change similar to Katsuno and Mendelzon on cyclic orders, we need a notion of closeness. We know that for any element $\mathrm{e} \in \mathrm{G}$ of a cyclically ordered set $(\mathrm{G}, \mathcal{C})$ the binary relation $\mathrm{x} <_{\mathrm{e},\mathcal{C}} \mathrm{y} \Leftrightarrow (\mathrm{e}, \mathrm{x}, \mathrm{y}) \in \mathcal{C}$ or $\mathrm{e} = \mathrm{x} \neq \mathrm{y}$ is a strict partial order. In this binary relation $<_{\mathrm{e},\mathcal{C}}$ e is minimal and every subset H of G has a set of minimal elements $\mathrm{Min}(\mathrm{H}, <_{\mathrm{e},\mathcal{C}})$ regarding this binary order (i.e. the elements of the subset that are closest to e). If $|\mathrm{Mod}(\psi)| = 1$ we could define such $<_{\omega_\psi,\mathcal{C}_\psi}$. However in general $|\mathrm{Mod}(\psi)| \neq 1$ and therefore we cannot use it. Instead we want a similar notion of closeness, that would also work in this special case. When we consider the general case with multiple $\omega_{\psi_\mathrm{i}} \in \mathrm{Mod}(\psi)$, each of these ω_{ψ_i} defines an individual binary order $<_{\omega_{\psi_\mathrm{i}},\mathcal{C}_\psi}$ and corresponding minimal sets for the set of possible worlds $\mathrm{Mod}(\mu)$ of a given propositional formula $\mu \in \mathcal{L}$. We assume that we can remember the set $\mathrm{Mod}(\psi)$ after the construction of a cyclic order $\mathcal{C}_\psi$ and this leaves us multiple possible combinations of these minimal sets. Of these possible combinations we consider the two most obvious ones:

1. $\omega \in \mathrm{Mod}(\mu)$ is 'minimal' in $\mathrm{Mod}(\mu)$ with respect to $\mathcal{C}_\psi$ and $\mathrm{Mod}(\psi)$ if it is minimal in $\mathrm{Mod}(\mu)$ with respect to every $<_{\omega_{\psi_\mathrm{i}},\mathcal{C}_\psi}$ (the binary order constructed with $\omega_{\psi_\mathrm{i}} \in \mathrm{Mod}(\psi)$ and $\mathcal{C}_\psi$).
2. $\omega \in \mathrm{Mod}(\mu)$ is 'minimal' in $\mathrm{Mod}(\mu)$ with respect to $\mathcal{C}_\psi$ and $\mathrm{Mod}(\psi)$ if it is minimal in $\mathrm{Mod}(\mu)$ with respect to at least one $<_{\omega_\psi,\mathcal{C}_\psi}$ (the binary order constructed with $\omega_\psi \in \mathrm{Mod}(\psi)$ and $\mathcal{C}_\psi$)

While they look very similar, the following example shows some of the problems of the first variant:

Example 4. *For a belief base ψ with $\mathrm{Mod}(\psi) = \{\omega_{\psi_1}, \omega_{\psi_2}, \omega_{\psi_3}\}$ and a propositional formula μ with $\mathrm{Mod}(\mu) = \{\omega_1, \omega_2, \omega_3\}$ we consider the cyclic order $\mathcal{C}_\psi$ which we receive from the set $\{(\omega_{\psi_1}, \omega_1, \omega_3), (\omega_{\psi_2}, \omega_2, \omega_1), (\omega_{\psi_3}, \omega_3, \omega_2)\}$. We can see that a cyclic order can be received from this set, because none of the three ternary-tuples share two elements (pairwise). Therefore we cannot receive another ternary-tuple through transitivity and:*

$$\begin{aligned}\mathcal{C}_\psi = \{&(\omega_{\psi_1}, \omega_1, \omega_3), (\omega_1, \omega_3, \omega_{\psi_1}), (\omega_3, \omega_{\psi_1}, \omega_1),\\ &(\omega_{\psi_2}, \omega_2, \omega_1), (\omega_2, \omega_1, \omega_{\psi_2}), (\omega_1, \omega_{\psi_2}, \omega_2),\\ &(\omega_{\psi_3}, \omega_3, \omega_2), (\omega_3, \omega_2, \omega_{\psi_3}), (\omega_2, \omega_{\psi_3}, \omega_3)\}\end{aligned}$$

We can define the three partial orders:

$$\begin{aligned}<_{\omega_{\psi_1},\mathcal{C}_\psi} &= \{(\omega_{\psi_1}, \omega_1), (\omega_{\psi_1}, \omega_3), (\omega_1, \omega_3)\},\\ <_{\omega_{\psi_2},\mathcal{C}_\psi} &= \{(\omega_{\psi_2}, \omega_2), (\omega_{\psi_2}, \omega_1), (\omega_2, \omega_1)\},\\ <_{\omega_{\psi_3},\mathcal{C}_\psi} &= \{(\omega_{\psi_3}, \omega_3), (\omega_{\psi_3}, \omega_2), (\omega_3, \omega_2)\}\end{aligned}$$

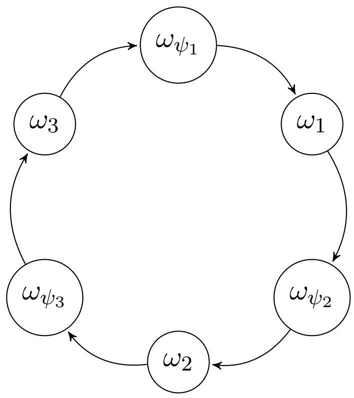

Figure 3: Possible cyclic order in example 4

We see that ω_1 *is not minimal in* $<_{\omega_{\psi_2},\mathcal{C}_\psi}$, *while* ω_2 *is not minimal in* $<_{\omega_{\psi_3},\mathcal{C}_\psi}$ *and* ω_3 *is not minimal in* $<_{\omega_{\psi_3},\mathcal{C}_\psi}$. *From our strict definition it follows that no model of* μ *is 'minimal' in* $\mathcal{C}_\psi$ *although* $\mathrm{Mod}(\mu) \neq \emptyset$ *and therefore* μ *is satisfiable and* $\psi \circ \mu$ *should be satisfiable. It follows that we cannot use this definition for a closeness relation, with the same function as the closeness relation of Katsuno Mendelzon on partial/pre-orders, on certain cyclic orders. At this point we have not yet defined the kind of cyclic orders that are permissable. We could formulate certain criteria that a cyclic order* $\mathcal{C}_\psi$ *has to fullfill to prevent such a scenario. However the following cyclic order entails this case: If we want to use the proposed strict version of closeness in cyclic orders our version of a faithful assignment would need to exclude such cyclic orders. However we want as few restrictions as possible. Therefore we choose the more lenient version for our notion of closeness in cyclic orders.*

For our purposes, we don't want to always define the binary orders and also avoid the use of the term 'minimal' in cyclic orders. Instead we want to emphasize the notion of *betweenness*, which is essential for the understanding of cyclic orders. Consequently we consider ω *close* to a set $\mathcal{M}^*$ with respect to $\mathcal{M}$ in the cyclic order $\mathcal{C}$ if there is some $\omega^* \in \mathcal{M}^*$ such that, starting from this ω^*, there is no $\tilde{\omega} \in \mathcal{M}$ *between* ω^* and ω. We formalize this in the following definition:

Definition 9. *An interpretation* $\omega \in \Omega$ *is called close to* $\mathcal{M}^* \subseteq \Omega$ *with respect to* $\mathcal{M} \subseteq \Omega$ *in a partial cyclic order* C *on* Ω *if and only if there exists some* $\omega^* \in \mathcal{M}^*$ *such that no* $\omega' \in \mathcal{M}$ *satisfies* $(\omega^*, \omega', \omega) \in C$. *We define*

$$\mathrm{closest}(\mathcal{M}, \mathcal{M}^*, \mathrm{C}) = \{\omega \in \mathcal{M} | \ \omega \text{ is close to } \mathcal{M}^* \text{ with respect to } \mathcal{M} \text{ in C}\}$$

The following lemma shows the connection between our definition of closeness and the minimality in the binary orders of $\mathrm{Mod}(\psi)$:

Lemma 8. *Let* $\mathcal{C}_\psi$ *be a cyclic order on* Ω *and* $\mu, \psi \in \mathcal{L}$, *then* $\omega \in \mathrm{Mod}(\mu)$ *is close to* $\mathrm{Mod}(\psi) \neq \emptyset$ *in* $\mathcal{C}_\psi$ *with respect to* $\mathrm{Mod}(\mu)$ *if and only if it is minimal in* $\mathrm{Mod}(\mu)$ *with respect to some* $<_{\omega_\psi,\mathcal{C}_\psi}$ *and* $\omega_\psi \in \mathrm{Mod}(\psi)$.

Proof. If ω is close to $\mathrm{Mod}(\psi)$ with respect to $\mathrm{Mod}(\mu)$ then there exists some $\omega_\psi \in \mathrm{Mod}(\psi)$ such that for all $\tilde{\omega} \in \mathrm{Mod}(\mu)$: $(\omega_\psi, \tilde{\omega}, \omega) \notin \mathcal{C}_\psi$. It follows that ω is minimal in $\mathrm{Mod}(\mu)$ with respect to $<_{\omega_\psi, \mathcal{C}_\psi}$.

If ω is minimal in $\mathrm{Mod}(\mu)$ with respect to some $<_{\omega_\psi, \mathcal{C}_\psi}$ and $\omega_\psi \in \mathrm{Mod}(\psi)$ it follows that for this ω_ψ there exists no $\tilde{\omega} \in \mathrm{Mod}(\mu)$ such that $(\omega_\psi, \tilde{\omega}, \omega) \in \mathcal{C}_\psi$ is satisfied. Then ω is close to $\mathrm{Mod}(\psi)$ with respect to $\mathrm{Mod}(\mu)$ in $\mathcal{C}_\psi$. □

3.2 Weak cyclic assignment

With our definition of closeness in cyclic orders, we do not have any initial restrictions on the set of cyclic orders on Ω, that we consider for cyclic revision. We need to emphasize that the set, for which we analyze the closeness of $\omega \in \mathrm{Mod}(\mu)$, must be nonempty. Otherwise the existential quantification cannot be satisfied. Moreover in order to satisfy (R4), this set and the cyclic orders must be equal for two equivalent knowledge bases. The following definition of a 'weak' cyclic assignment summarises these thoughts:

Definition 10. *A* 'weak' faithful cyclic assignment *is a function $\psi \mapsto (\mathcal{C}_\psi, \mathcal{M}_\psi)$ that assigns to every $\psi \in \mathcal{L}$ a partial cyclic order $\mathcal{C}_\psi \subseteq \Omega \times \Omega \times \Omega$ and $\mathcal{M}_\psi \subseteq \Omega$ such that:*

1. *If $\psi \equiv \phi$, then $\mathcal{C}_\psi = \mathcal{C}_\phi$ and $\mathcal{M}_\psi = \mathcal{M}_\phi$.*
2. *If $\mathrm{Mod}(\psi) \neq \emptyset$ then $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$, if not we demand $\mathcal{M}_\psi \neq \emptyset$.*

We call this a 'weak' faithful cyclic assignment because in definition 11 we define an alternative more strict version. The reason for that lies in the fact that we allow in the 'weak' case cyclic orders that do not satisfy postulates which ensure that the orders are *well behaved* (see [KM91]). At this point we want to emphasize that the definition of a weak faithful cyclic assignment does indeed allow any possible cyclic order on Ω. For a cyclic order $\mathcal{C}$ on Ω we can define a trivial function:

$$\mathrm{f} : \mathcal{L} \to \mathcal{P}(\Omega \times \Omega \times \Omega) \times \mathcal{P}(\Omega);\ \psi \mapsto (\mathcal{C}, \Omega)$$

The only case in which this function is not a weak faithful cyclic assignment is if $\Omega = \emptyset$. However in this case the language would be empty and there would be no point in revising no beliefs with no beliefs. Therefore it would be excessive to demand that $\Omega \neq \emptyset$.

3.3 Katsuno Mendelzon Postulates and cyclic revision

While we have defined a closeness operator on cyclic orders, it cannot directly take the place of the closeness operator on partial orders defined by Katsuno and Mendelzon. The reason for that lies in the case $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$. Here the result of $\psi \circ \mu$ should be $\psi \wedge \mu$ because of (R2). If we would simply set $\mathrm{Mod}(\psi \circ \mu) = \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi)$ the resulting change-operator $\circ$ could violate (R2):

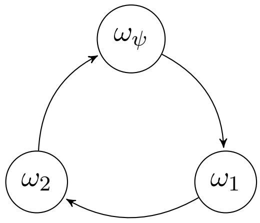

Figure 4: Cyclic order in example 5

Example 5. *Let* $\mathrm{Mod}(\mu) = \{\omega_\psi, \omega_1\}$ *and* $\mathrm{Mod}(\psi) = \{\omega_\psi\}$*. Because* $\mathrm{Mod}(\psi) \neq \emptyset$ *it follows that* $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$*. In the cyclic order* $\mathcal{C}_\psi = \{(\omega_\psi, \omega_1, \omega_2)\}$ *we can see that starting from* $\omega_\psi \in \mathrm{Mod}(\psi)$ *there is no* $\tilde{\omega} \in \mathrm{Mod}(\mu)$ *between* ω_ψ *and* ω_1*. Therefore* ω_1 *is close to* $\mathrm{Mod}(\psi)$ *with respect to* $\mathrm{Mod}(\mu)$ *in* $\mathcal{C}_\psi$ *and* $\mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) = \{\omega_\psi, \omega_1\}$*. However* $\omega_1 \notin \mathrm{Mod}(\psi \wedge \mu)$ *even though* $\psi \wedge \mu$ *is satisfiable and thus* $\psi \circ \mu \equiv \psi \wedge \mu$ *should be the case.*

To ensure (R2) we therefore simply demand in the case $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$ that $\mathrm{Mod}(\psi \circ \mu) = \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu)$.

We have defined a belief-change operator on cyclic orders and can now analyse which postulates are satisfied by this operator. In the following theorem we formally define this operator and show that it satisfies (R1) to (R4):

Theorem 9. *Let* $\psi \in \mathcal{L}$ *be a knowledge base,* $\mu \in \mathcal{L}$ *a piece of new information,* $\mathcal{C}_\psi$ *a faithful cyclic order on* Ω *and* $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$ *if* $\mathrm{Mod}(\psi) \neq \emptyset$ *else* $\emptyset \neq \mathcal{M}_\psi \subseteq \Omega$*. The belief-change operator*

$$\mathrm{Mod}(\psi \circ \mu) = \begin{cases} \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) & \textit{if } \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \textit{else} \end{cases}$$

satisfies the basic postulates (R1) to (R4).

Proof. We show these postulates in linear order:
(R1) In the case $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$ then $\mathrm{Mod}(\psi \circ \mu) = \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \subseteq \mathrm{Mod}(\mu)$ and therefore $\psi \circ \mu$ implies μ. If $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$ then $\mathrm{Mod}(\psi \circ \mu) = \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi)$ which means all $\omega \in \mathrm{Mod}(\psi \circ \mu)$ are in $\mathrm{Mod}(\mu)$ and close to $\mathcal{M}_\psi$ with respect to $\mathrm{Mod}(\mu)$ in $\mathcal{C}_\psi$. That means (R1) is satisfied.
(R2) If $\psi \wedge \mu$ is satisfiable then $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$ and therefore $\mathrm{Mod}(\psi \circ \mu) = \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu)$ and with that $\psi \circ \mu \equiv \psi \wedge \mu$
(R3) If $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$ then $\mathrm{Mod}(\psi \circ \mu) = \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$ and with that $\psi \circ \mu$ is satisfiable. In the other case we assume towards contradiction that $\mathrm{Mod}(\mu) \neq \emptyset$ and $\mathrm{Mod}(\psi \circ \mu) = \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) = \emptyset$, i.e. $\psi \circ \mu$ is not satisfiable. Then all $\omega \in \mathrm{Mod}(\mu)$ are not close to $\mathcal{M}_\psi$ with respect to $\mathrm{Mod}(\mu)$ in

$\mathcal{C}_\psi$. Because $\mathcal{M}_\psi \neq \emptyset$ for $\omega \in \mathrm{Mod}(\mu)$ and $\omega_\psi \in \mathcal{M}_\psi$ it follows that there exists $\tilde{\omega} \in \mathrm{Mod}(\mu)$ such that $(\omega_\psi, \tilde{\omega}, \omega) \in \mathcal{C}_\psi$. However this $\tilde{\omega}$ cannot be close itself and therefore another $\hat{\omega} \in \mathrm{Mod}(\mu)$ has to exist with $(\omega_\psi, \hat{\omega}, \tilde{\omega}) \in \mathcal{C}_\psi$. Because $\mathcal{C}_\psi$ is transitive and asymmetric it follows that $(\omega_\psi, \hat{\omega}, \omega) \in \mathcal{C}_\psi$ and $\hat{\omega} \neq \omega$. We define $\mathcal{M}^3_\mu = \{\omega, \tilde{\omega}, \hat{\omega}\}$ and because $\hat{\omega}$ is not close there exists $\omega_4 \in \mathrm{Mod}(\mu)$ such that $(\omega_\psi, \omega_4, \hat{\omega}) \in \mathcal{C}_\psi$ and for this ω_4 there exists $\omega_5 \in \mathrm{Mod}(\mu)$ and so on and so forth. For $\mathcal{M}^{\mathrm{i}}_\mu \subseteq \mathrm{Mod}(\mu)$, $\omega_{\mathrm{i}+1} \in \mathrm{Mod}(\mu)$ and $\omega_{\mathrm{i}} \in \mathrm{Min}(\mathcal{M}^{\mathrm{i}}_\mu)$ with $(\omega_\psi, \omega_{\mathrm{i}+1}, \omega_{\mathrm{i}}) \in \mathcal{C}_\psi$ then because of transitivity $\omega_{\mathrm{i}+1} \in \mathrm{Min}(\mathcal{M}^{\mathrm{i}+1}_\mu = \mathcal{M}^{\mathrm{i}}_\mu \cup \{\omega_{\mathrm{i}+1}\})$ and because of asymmetry $\omega_{\mathrm{i}+1} \notin \mathcal{M}^{\mathrm{i}}_\mu$. Because $\mathcal{L}$ and Ω are finite $\mathrm{Mod}(\mu)$ is finite. Therefore there is a final iteration $\mathcal{M}^{\mathrm{i}}_\mu$ of $\mathcal{M}^3_\mu$ and there exists $\omega_{\mathrm{i}} \in \mathrm{Min}(\mathcal{M}^{\mathrm{i}}_\mu, <_{\omega_\psi})$ and this ω_i is close to $\mathcal{M}_\psi$ with respect to $\mathrm{Mod}(\mu)$ in $\mathcal{C}_\psi$ which is a contradiction and therefore $\psi \circ \mu$ is satisfiable if μ is satisfiable.
(R4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$ because of condition 1 of definition 10 it follows that $\mathcal{C}_{\psi_1} = \mathcal{C}_{\psi_2}$ and $\mathcal{M}_{\psi_1} = \mathcal{M}_{\psi_2}$. Since $\mu_1 \equiv \mu_2$ means $\mathrm{Mod}(\mu_1) = \mathrm{Mod}(\mu_2)$ and therefore $\psi_1 \circ \mu_1 \equiv \psi_2 \circ \mu_2$. $\square$

We show next that our definition of closeness on faithful cyclic orders, like the definition of closeness on faithful partial orders [KM91], does satisfy (R5) but not (R6):

Theorem 10. *Let $\psi \in \mathcal{L}$ be a knowledge base, $\mu \in \mathcal{L}$ a piece of new information, $\mathcal{C}_\psi$ a faithful cyclic order on Ω and $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$ if $\mathrm{Mod}(\psi) \neq \emptyset$ else $\emptyset \neq \mathcal{M}_\psi \subseteq \Omega$. The belief-change operator*

$$\mathrm{Mod}(\psi \circ \mu) = \begin{cases} \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) & \textit{if } \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \textit{else} \end{cases}$$

satisfies postulate (R5) but not (R6).

Proof. For (R5) we distinguish the two cases:

1. If $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$ then $\psi \circ \mu \equiv \psi \wedge \mu$ and therefore if $\mathrm{Mod}(\psi \wedge \mu) \cap \mathrm{Mod}(\phi) \neq \emptyset$ then $(\psi \circ \mu) \wedge \phi \equiv \psi \wedge \mu \wedge \phi \equiv \psi \circ (\mu \wedge \phi)$. Otherwise if $\mathrm{Mod}(\psi \wedge \mu) \cap \mathrm{Mod}(\phi) = \emptyset$ then $(\psi \circ \mu) \wedge \phi$ is not satisfiable and therefore implies $\psi \circ (\mu \wedge \phi)$.

2. If $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) = \emptyset$ then $\mathrm{Mod}(\psi \circ \mu) = \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi)$. If $\mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) \cap \mathrm{Mod}(\phi) = \emptyset$ then $(\psi \circ \mu) \wedge \phi$ implies $\psi \circ (\mu \wedge \phi)$. Let $\omega \in \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) \cap \mathrm{Mod}(\phi) \neq \emptyset$ then there exists an $\omega_\psi \in \mathcal{M}_\psi$ such that for all $\tilde{\omega} \in \mathrm{Mod}(\mu) :\ (\omega_\psi, \tilde{\omega}, \omega) \notin \mathcal{C}_\psi$ and $\omega \in \mathrm{Mod}(\phi)$. Towards contradiction we assume that $\omega \notin \mathrm{closest}(\mathrm{Mod}(\mu \wedge \phi), \mathcal{M}_\psi, \mathcal{C}_\psi)$, then it follows that for all $\hat{\omega}_\psi \in \mathcal{M}_\psi$ there exists an $\tilde{\omega} \in \mathrm{Mod}(\mu \wedge \phi)$ such that $(\hat{\omega}_\psi, \tilde{\omega}, \omega) \in \mathcal{C}_\psi$. That means that there exists an $\tilde{\omega} \in \mathrm{Mod}(\mu \wedge \phi)$ such that $(\omega_\psi, \tilde{\omega}, \omega) \in \mathcal{C}_\psi$, however $\tilde{\omega} \in \mathrm{Mod}(\mu \wedge \phi) = \mathrm{Mod}(\mu) \cap \mathrm{Mod}(\phi) \subseteq \mathrm{Mod}(\mu)$ which is a contradiction. Therefore $\omega \in \mathrm{closest}(\mathrm{Mod}(\mu \wedge \phi), \mathcal{M}_\psi, \mathcal{C}_\psi)$, which means that $(\mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) \cap \mathrm{Mod}(\phi)) \subseteq \mathrm{closest}(\mathrm{Mod}(\mu \wedge \phi), \mathcal{M}_\psi, \mathcal{C}_\psi)$ i.e. $(\psi \circ \mu) \wedge \phi$ implies $\psi \circ (\mu \wedge \phi)$.

For (R6) we use the example of a revision operator that satisfies (R1) to (R5), (R7) and (R8), but not (R6) by Katsuno and Mendelzon (example 5.4. in [KM91]):

Let each $\mathrm{a} \in \Sigma$ have two weights $\mathrm{w_a}, \hat{\mathrm{w}}_\mathrm{a} \in \mathbb{R}$ each representing some measure of likelihood that the truth value of a will be changed. For two possible worlds ω_i and ω_j, we denote by $\mathrm{Diff}(\omega_\mathrm{i}, \omega_\mathrm{j})$ the set of propositional letters whose interpretation is different in ω_i and ω_j. For $\mu \in \mathcal{L}$ we denote by $\mathrm{Diff}(\omega_\mathrm{i}, \mu)$ the collection of $\mathrm{Diff}(\omega_\mathrm{i}, \omega_\mathrm{j})$ where $\omega_\mathrm{j} \in \mathrm{Mod}(\mu)$.

We define two distances between two possible worlds:

$$\mathrm{dist}_1(\omega_1, \omega_2) = \sum_{\mathrm{x} \in \mathrm{Diff}(\omega_1, \omega_2)} \mathrm{w_x},$$

$$\mathrm{dist}_2(\omega_1, \omega_2) = \sum_{\mathrm{x} \in \mathrm{Diff}(\omega_1, \omega_2)} \hat{\mathrm{w}}_\mathrm{x}$$

and the distances between $\mathrm{Mod}(\psi)$ and $\omega \in \Omega$:

$$\mathrm{dist}_1(\psi, \omega) = \min_{\omega_\psi \in \mathrm{Mod}(\psi)} \mathrm{dist}_1(\omega_\psi, \omega),$$

$$\mathrm{dist}_1(\psi, \omega) = \min_{\omega_\psi \in \mathrm{Mod}(\psi)} \mathrm{dist}_1(\omega_\psi, \omega).$$

At this point Katsuno and Mendelzon define a partial pre-order for each knowledge base ψ as: " $I \leqslant_\psi I^{'}$ if and only if $dist_1(\psi, I) \leqslant dist_1(\psi, I)$ and $dist_2(\psi, I) \leqslant dist_2(\psi, I)$" [KM91]. Because Katsuno and Mendelzon argue that "it is easy to show that $\leqslant_\psi$ is actually a partial pre-order and is not a toal pre-order in general" [KM91], we assume that they made a mistake and meant: $\mathrm{I} \leqslant_\psi \mathrm{I}^{'}$ if and only if $\mathrm{dist}_1(\psi, \mathrm{I}) \leqslant \mathrm{dist}_1(\psi, \mathrm{I}^{'})$ and $\mathrm{dist}_2(\psi, \mathrm{I}) \leqslant \mathrm{dist}_2(\psi, \mathrm{I}^{'})$. Because $\leqslant_\psi$ is a partial pre-order the revision operator defined as

$$\mathrm{Mod}(\psi \circ \mu) = \mathrm{Min}(\mathrm{Mod}(\mu), \leqslant_\psi)$$

satisfies (R1) to (R5), (R7) and (R8).

We consider the case by Katsuno and Mendelzon where (R6) does not hold: Let

$$\langle \mathrm{w_a}, \mathrm{w_b}, \mathrm{w_c}, \mathrm{w_d} \rangle = \langle 1.0, 2.0, 3.0, 4.0 \rangle,$$

$$\langle \hat{\mathrm{w}}_\mathrm{a}, \hat{\mathrm{w}}_\mathrm{b}, \hat{\mathrm{w}}_\mathrm{c}, \hat{\mathrm{w}}_\mathrm{d} \rangle = \langle 4.0, 3.0, 2.0, 1.0 \rangle.$$

and the possible worlds

$$\omega_{\psi_1} = \langle 1, 1, 1, 1 \rangle,\ \omega_{\psi_2} = \langle 0, 0, 0, 0 \rangle$$

$$\omega_1 = \langle 0, 0, 1, 1 \rangle,\ \omega_2 = \langle 1, 0, 0, 0 \rangle,\ \omega_3 = \langle 0, 0, 1, 0 \rangle.$$

We obtain

$$\mathrm{dist}_1(\psi, \omega_1) = 3,\ \mathrm{dist}_2(\psi, \omega_1) = 3$$

$$\mathrm{dist}_1(\psi, \omega_2) = 1,\ \mathrm{dist}_2(\psi, \omega_2) = 4$$

$$\mathrm{dist}_1(\psi, \omega_3) = 3, \; \mathrm{dist}_2(\psi, \omega_3) = 2$$

and can build a cyclic order $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_3, \omega_1), (\omega_{\psi_2}, \omega_3, \omega_1)\}$.

Suppose $\psi = \mathrm{form}(\omega_{\psi_1}, \omega_{\psi_2})$ and $\mathcal{M}_\psi = \mathrm{Mod}(\psi) \neq \emptyset$, $\mu = \mathrm{form}(\omega_1, \omega_2, \omega_3)$ and $\phi = \mathrm{form}(\omega_1, \omega_2)$. Then, because $\emptyset = \mathrm{Mod}(\mu) \cap \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\phi) = \emptyset$,

$$\mathrm{Mod}(\psi \circ \mu) = \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) = \{\omega_2, \omega_3\}$$

and

$$\mathrm{Mod}(\psi \circ \phi) = \mathrm{closest}(\mathrm{Mod}(\phi), \mathcal{M}_\psi, \mathcal{C}_\psi) = \{\omega_1, \omega_2\}.$$

We see that $((\psi \circ \mu) \wedge \phi) \equiv \mathrm{form}(\omega_2)$ and $(\psi \circ (\mu \wedge \phi)) \equiv \mathrm{form}(\omega_1, \omega_2)$ does not imply $(\psi \circ \mu) \wedge \phi$, i.e. $\psi \circ (\mu \wedge \phi)$ does not imply $(\psi \circ \mu) \wedge \phi$. In other words (R6) does not hold [KM91]. □

Lemma 11. *Let* $\omega_\psi \in \mathcal{M}_\psi \neq \emptyset$, $\mathrm{Mod}(\mu) \neq \emptyset$ *and* $\omega \in \mathrm{Mod}(\mu)$. *If there exists* $\omega_1 \in \mathrm{Mod}(\mu)$ *such that* $(\omega_\psi, \omega_1, \omega) \in \mathcal{C}_\psi$ *then either* ω_1 *is close to* $\mathcal{M}_\psi$ *with respect to* $\mathrm{Mod}(\mu)$ *in* $\mathcal{C}_\psi$*, or there exists* $\omega_2 \in \mathrm{Mod}(\mu)$ *such that* ω_2 *is close and* $(\omega_\psi, \omega_2, \omega_1) \in \mathcal{C}_\psi$.

Proof. We define the strict partial order $<_{\omega_\psi, \mathcal{C}_\psi} = \{(\omega_\mathrm{i}, \omega_\mathrm{j}) | \, (\omega_\psi, \omega_\mathrm{i}, \omega_\mathrm{j}) \in \mathcal{C}_\psi\}$ which is finite because $\mathcal{C}_\psi$ is finite. Because $\omega_\psi \in \mathrm{Mod}(\mu)$ is possible we look at $<'_{\omega_\psi, \mathcal{C}_\psi} = <_{\omega_\psi, \mathcal{C}_\psi} \setminus \{(\omega_\psi, \omega_\mathrm{i}) \in <_{\omega_\psi, \mathcal{C}_\psi} | \, \omega_\mathrm{i} \in \Omega\}$. Because $<'_{\omega_\psi, \mathcal{C}_\psi}$ is finite and $\omega_1 <'_{\omega_\psi, \mathcal{C}_\psi} \omega$ it follows that $\mathrm{Min}(\mathrm{Mod}(\mu), <'_{\omega_\psi, \mathcal{C}_\psi}) \neq \emptyset$. Then either $\omega_1 \in \mathrm{Min}(\mathrm{Mod}(\mu), <'_{\omega_\psi, \mathcal{C}_\psi}$ and therefore ω_2 close to $\mathcal{M}_\psi$ or there exists $\omega_2 \in \mathrm{Min}(\mathrm{Mod}(\mu), <'_{\omega_\psi, \mathcal{C}_\psi}$. Then ω_2 is close to $\mathcal{M}_\psi$, $\omega_2 <'_{\omega_\psi, \mathcal{C}_\psi} \omega_1$ and therefore $(\omega_\psi, \omega_2, \omega_1) \in \mathcal{C}_\psi$ □

Theorem 12. *Let* $\psi \in \mathcal{L}$ *be a knowledge base,* $\mu \in \mathcal{L}$ *a piece of new information,* $\mathcal{C}_\psi$ *a faithful cyclic order on* Ω *and* $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$ *if* $\mathrm{Mod}(\psi) \neq \emptyset$ *else* $\emptyset \neq \mathcal{M}_\psi \subseteq \Omega$. *The belief-change operator*

$$\mathrm{Mod}(\psi \circ \mu) = \begin{cases} \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) & \textit{if}\, \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \textit{else} \end{cases}$$

satisfies postulate (R7) but not (R8).

Proof. For (R7) we consider the simple cases first: If $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_1) \neq \emptyset$, $\psi \circ \mu_1$ implies μ_2 and $\psi \circ \mu_2$ implies μ_1, then $\psi \circ \mu_1 \equiv \psi \wedge \mu_1$ and it follows that $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_2) \neq \emptyset$ which implies $\psi \circ \mu_2 \equiv \psi \wedge \mu_2$. In summary $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_1) \subseteq \mathrm{Mod}(\mu_2)$ and $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_2) \subseteq \mathrm{Mod}(\mu_1)$ implies $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_1) = \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_2)$ and therefore $\psi \circ \mu_1 \equiv \psi \circ \mu_2$. If μ_1 is not satisfiable then $\mathrm{Mod}(\mu_1) = \emptyset = \mathrm{Mod}(\psi \circ \mu_1) \subseteq \mathrm{Mod}(\mu_2)$ and because $\psi \circ \mu_2$ implies μ_1, which means $\mathrm{Mod}(\psi \circ \mu_2) \subseteq \emptyset$ and therefore $\psi \circ \mu_2$ is not satisfiable, which is only possible if μ_2 is not satisfiable and therefore $\mu_1 \equiv \mu_2$.

Let $\mathrm{Mod}(\mu_1) \neq \emptyset = \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_1)$ and let $\mathrm{Mod}(\psi \circ \mu_1) \subseteq \mathrm{Mod}(\mu_2)$ as well as $\mathrm{Mod}(\psi \circ \mu_2) \subseteq \mathrm{Mod}(\mu_1)$. Towards contradiction we assume that $\psi \circ \mu_1 \not\equiv \psi \circ \mu_2$, which means $\mathrm{Mod}(\psi \circ \mu_1) \neq \mathrm{Mod}(\psi \circ \mu_2)$ i.e.

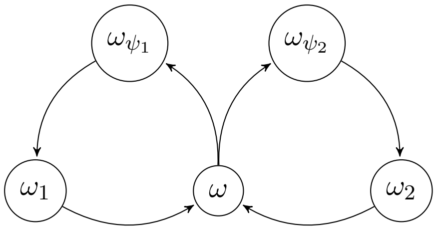

Figure 5: Cyclic order in example 6

$\mathrm{Mod}(\psi \circ \mu_1) \not\subseteq \mathrm{Mod}(\psi \circ \mu_2)$ or $\mathrm{Mod}(\psi \circ \mu_2) \not\subseteq \mathrm{Mod}(\psi \circ \mu_1)$. Without loss of generality it follows that there exists $\omega \in \mathrm{Mod}(\psi \circ \mu_1)$ such that $\omega \notin \mathrm{Mod}(\psi \circ \mu_2)$. Because $\psi \circ \mu_1$ implies μ_2 it follows that $\omega \in \mathrm{Mod}(\mu_2)$ and there exists some $\omega_\psi \in \mathcal{M}_\psi$ such that for all $\omega_1 \in \mathrm{Mod}(\mu_1) : (\omega_\psi, \omega_1, \omega) \notin \mathcal{C}_\psi$. Since $\omega \notin \mathrm{Mod}(\psi \circ \mu_2)$ there exists an $\omega_2 \in \mathrm{Mod}(\mu_2)$ such that $(\omega_\psi, \omega_2, \omega) \in \mathcal{C}_\psi$. If ω_2 is not close to $\mathcal{M}_\psi$, we can infer with lemma 11 that there exists some $\tilde{\omega}_2 \in \mathrm{Mod}(\mu_2)$ that is close to $\mathcal{M}_\psi$ and because of transitivity $(\omega_\psi, \tilde{\omega}_2, \omega_2) \in \mathcal{C}_\psi$. Because of transitivity it follows that $(\omega_\psi, \tilde{\omega}_2, \omega_2) \in \mathcal{C}_\psi$ and $\tilde{\omega}_2 \in \mathrm{Mod}(\mu_1)$ because $\psi \circ \mu_2$ implies μ_1. This is a contradiction and therefore $\omega \in \mathrm{Mod}(\psi \circ \mu_2)$ and consequently $\psi \circ \mu_1 \equiv \psi \circ \mu_2$.
For (R8) the central counterexample consists of the cyclic order $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_2}, \omega_2, \omega)\}$. We will study this counterexample separately in example 6. $\square$

The following example shows a cyclic order that does not satisfy postulate (R8):

Example 6. *Let* $\mathrm{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_2}, \omega_2, \omega)\}$, $\mathrm{Mod}(\psi) = \{\omega_{\psi_1}, \omega_{\psi_2}\}$, $\mathrm{Mod}(\mu_1) = \{\omega_1, \omega\}$ *and* $\mathrm{Mod}(\mu_2) = \{\omega_2, \omega\}$. *We see that* $\mathrm{Mod}(\psi \circ \mu_1) = \{\omega_1, \omega\}$ *because* $(\omega_{\psi_1}, \omega, \omega_1) \notin \mathcal{C}_\psi$ *and* $(\omega_{\psi_2}, \omega_1, \omega) \notin \mathcal{C}_\psi$, *also* $\mathrm{Mod}(\psi \circ \mu_2) = \{\omega_2, \omega\}$ *because* $(\omega_{\psi_2}, \omega, \omega_2) \notin \mathcal{C}_\psi$ *and* $(\omega_{\psi_1}, \omega_2, \omega) \notin \mathcal{C}_\psi$. *Therefore* $\mathrm{Mod}(\psi \circ \mu_1 \wedge \psi \circ \mu_2) = \mathrm{Mod}(\psi \circ \mu_1) \cap \mathrm{Mod}(\psi \circ \mu_2) = \{\omega\}$ *but* $\mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2)) = \{\omega_1,\ \omega_2\}$ *because* $(\omega_{\psi_1}, \omega_1, \omega) \in \mathcal{C}_\psi$ *and* $(\omega_{\psi_2}, \omega_2, \omega) \in \mathcal{C}_\psi$. *That means* $\psi \circ \mu_1 \wedge \psi \circ \mu_2$ *does not imply* $\psi \circ (\mu_1 \vee \mu_2)$.

Because we allow faithful cyclic assignments that map $\psi \in \mathcal{L}$ to cyclic orders on Ω, that together with our notion of closeness and the resulting belief-change-operator, do not satisfy (R8), we need other ways to achieve a representation-theorem. For this we have two options:

1. We can consider more strict definitions for closeness, or the faithful cyclic assignment in order to satisfy (R8).

2. We can formulate new additional postulates.

In the next section we show that a stricter version of the faithful cyclic assignment is sufficient to satisfy (R8).

3.4 Strong cyclic assignment

Before we can remedy the problem presented in example 6, we need to understand it. While ω is not in $\mathrm{Min}(\mathrm{Mod}(\mu_1), <_{\psi_1,\mathcal{C}_\psi})$ it is in $\mathrm{Min}(\mathrm{Mod}(\mu_1), <_{\psi_2,\mathcal{C}_\psi})$ and for μ_2 the other way around. Our notion of closeness is equivalent to ω being minimal in at least one of the $\omega_\psi \in \mathcal{C}_\psi : <_{\omega_\psi,\mathcal{C}_\psi}$. The stricter version, meaning ω needing to be minimal in all of these partial orders, would rectify this case. However the stricter version forces a stronger version of the cyclic assignment, otherwise (R2) would not be satisfied (for example $\mathrm{Mod}(\psi) = \{\omega_{\psi_1}, \omega_{\psi_2}\}$, $\mathrm{Mod}(\mu) = \{\omega_1, \omega_2\}$ and $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega_2), (\omega_{\psi_2}, \omega_2, \omega_1)\}$). Therefore it is advisable to first look at the faithful cyclic assignment itself. While the condition

$$\text{if } \omega_\psi \in \mathcal{M}_\psi,\ \omega_1, \omega_2 \in \Omega\backslash\mathcal{M}_\psi \text{ with } (\omega_\psi, \omega_1, \omega_2) \in \mathcal{C}_\psi \text{ then for all}$$
$$\tilde{\omega}_\psi \in \mathcal{M}_\psi :\ (\tilde{\omega}_\psi, \omega_1, \omega_2) \in \mathcal{C}_\psi$$

certainly ensures (R8), because the cyclic order would essentially be a Katsuno and Mendelzon partial order converted into a cyclic order, it is too strict because it would prohibit the cyclic order of example 4. Instead we add a condition that allows such cyclic orders and call this version a *strong* faithful cyclic assignment:

Definition 11. *A* strong faithful cyclic assignment *is a function $\psi \mapsto (C_\psi, \mathcal{M}_\psi)$ that assigns to every $\psi \in \mathcal{L}$ a partial cyclic order $C_\psi \subseteq \Omega \times \Omega \times \Omega$ and $\mathcal{M}_\psi \subseteq \Omega$ such that:*

1. *If $\psi \equiv \phi$, then $C_\psi = C_\phi$ and $\mathcal{M}_\psi = \mathcal{M}_\phi$.*
2. $\mathcal{M}_\psi \neq \emptyset$
3. *If $\omega_\psi \in \mathcal{M}_\psi$ and $(\omega_\psi, \omega_1, \omega_2) \in C_\psi$ then for all $\tilde{\omega}_\psi \in \mathcal{M}_\psi,\ \tilde{\omega}_\psi \neq \omega_i\ for\ i \in \{1, 2\}$, either $(\tilde{\omega}_\psi, \omega_1, \omega_2) \in C_\psi$ or $(\tilde{\omega}_\psi, \omega_2, \omega_1) \in C_\psi$.*

Accordingly we call the faithful cyclic assignments of definition 10 *weak*. To show the impact of this additional condition we revisit example 6:

Example 7. *The cyclic order $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_2}, \omega_2, \omega)\}$, $\mathrm{Mod}(\psi) = \{\omega_{\psi_1}, \omega_{\psi_2}\}$, $\mathrm{Mod}(\mu_1) = \{\omega_1, \omega\}$ and $\mathrm{Mod}(\mu_2) = \{\omega_2, \omega\}$ is not a* strong *faithful cyclic order because neither $(\omega_{\psi_2}, \omega_1, \omega) \in \mathcal{C}_\psi$ nor $(\omega_{\psi_2}, \omega, \omega_1) \in \mathcal{C}_\psi$. There are four possible combinations to elevate $\mathcal{C}_\psi$ into a strong faithful cyclic order:*

1. *Let $\mathcal{C}'_\psi = \mathcal{C}_\psi \cup \{(\omega_{\psi_1}, \omega_2, \omega), (\omega_{\psi_2}, \omega_1, \omega)\}$. Then $\mathcal{C}'_\psi$ is a cyclic order because there is no transitive case. When we look at the belief-change we receive $\mathrm{Mod}(\psi \circ \mu_1) = \{\omega_1\}$ and $\mathrm{Mod}(\psi \circ \mu_2) = \{\omega_2\}$. Therefore $\mathrm{Mod}((\psi \circ \mu_1) \wedge (\psi \circ \mu_2)) = \emptyset \subseteq \{\omega_1, \omega_2\} = \mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2))$ i.e. $(\psi \circ \mu_1) \wedge (\psi \circ \mu_2)$ implies $\psi \circ (\mu_1 \vee \mu_2)$. Two possible cyclic orders:*
2. *Let $\mathcal{C}'_\psi = \mathcal{C}_\psi \cup \{(\omega_{\psi_1}, \omega_2, \omega), (\omega_{\psi_2}, \omega, \omega_1)\}$. Then $\mathcal{C}'_\psi$ is not a cyclic order because there are transitive cases. We list these cases and their implications:*
    a) $(\omega_{\psi_2}, \omega_2, \omega) \in \mathcal{C}_\psi$ and $(\omega_{\psi_2}, \omega, \omega_1) \in \mathcal{C}_\psi$ *implies* $(\omega_{\psi_2}, \omega_2, \omega_1) \in \mathcal{C}_\psi$

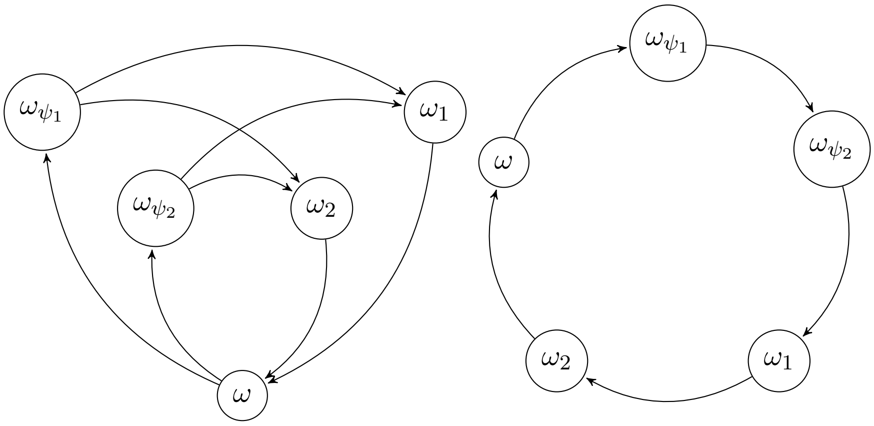

Figure 6: Two possible cyclic orders for case 1 in example 7

b) $(\omega_{\psi_1}, \omega_1, \omega) \in \mathcal{C}_\psi$ and $(\omega_{\psi_2}, \omega, \omega_1) \in \mathcal{C}_\psi$ *implies* $(\omega_{\psi_1}, \omega_1, \omega_{\psi_2}) \in \mathcal{C}_\psi$ *and* $(\omega_{\psi_2}, \omega, \omega_{\psi_1}) \in \mathcal{C}_\psi$

c) $(\omega_{\psi_2}, \omega_2, \omega) \in \mathcal{C}_\psi$ and $(\omega_{\psi_2}, \omega, \omega_1) \in \mathcal{C}_\psi$ *implies* $(\omega_1, \omega_2, \omega) \in \mathcal{C}_\psi$

d) $(\omega_{\psi_2}, \omega_2, \omega) \in \mathcal{C}_\psi$ and $(\omega_{\psi_2}, \omega, \omega_{\psi_1}) \in \mathcal{C}_\psi$ *implies* $(\omega_{\psi_2}, \omega_2, \omega_{\psi_1}) \in \mathcal{C}_\psi$

e) $(\omega_{\psi_1}, \omega_1, \omega_{\psi_2}) \in \mathcal{C}_\psi$ and $(\omega_{\psi_1}, \omega_{\psi_2}, \omega_2) \in \mathcal{C}_\psi$ *implies* $(\omega_{\psi_1}, \omega_1, \omega_2) \in \mathcal{C}_\psi$

In total we can extend $\mathcal{C}'_\psi$ to a complete cyclic order: As we see the result of the revision with μ_1 is still $\mathrm{Mod}(\psi \circ \mu_1) = \{\omega_1, \omega\}$, however for μ_2 the result is $\mathrm{Mod}(\psi \circ \mu_2) = \{\omega_2\}$ and therefore $\mathrm{Mod}((\psi \circ \mu_1) \wedge (\psi \circ \mu_2)) = \emptyset$ implies $\psi \circ (\mu_1 \vee \mu_2)$.

3. *Let $\mathcal{C}'_\psi = \mathcal{C}_\psi \cup \{(\omega_{\psi_1}, \omega, \omega_2), (\omega_{\psi_2}, \omega_1, \omega)\}$. Then $\mathcal{C}'_\psi$ is not a cyclic order because there are transitive cases. We list these cases and their implications:*

   a) $(\omega_{\psi_1}, \omega_1, \omega) \in \mathcal{C}_\psi$ and $(\omega_{\psi_1}, \omega, \omega_2) \in \mathcal{C}_\psi$ *implies* $(\omega_{\psi_1}, \omega_1, \omega_2) \in \mathcal{C}_\psi$

   b) $(\omega_{\psi_1}, \omega_1, \omega) \in \mathcal{C}_\psi$ and $(\omega_{\psi_1}, \omega, \omega_2) \in \mathcal{C}_\psi$ *implies* $(\omega_2, \omega_1, \omega) \in \mathcal{C}_\psi$

   c) $(\omega_{\psi_1}, \omega, \omega_2) \in \mathcal{C}_\psi$ and $(\omega_{\psi_2}, \omega_2, \omega) \in \mathcal{C}_\psi$ *implies* $(\omega_{\psi_1}, \omega_{\psi_2}, \omega_2) \in \mathcal{C}_\psi$

   d) $(\omega_{\psi_2}, \omega_2, \omega) \in \mathcal{C}_\psi$ and $(\omega_2, \omega_1, \omega) \in \mathcal{C}_\psi$ *implies* $(\omega_{\psi_2}, \omega_2, \omega_1) \in \mathcal{C}_\psi$

   e) $(\omega_{\psi_1}, \omega_1, \omega_2) \in \mathcal{C}_\psi$ and $(\omega_{\psi_2}, \omega_2, \omega_1) \in \mathcal{C}_\psi$ *implies* $(\omega_{\psi_2}, \omega_{\psi_1}, \omega_1) \in \mathcal{C}_\psi$

   f) $(\omega_{\psi_2}, \omega_2, \omega_{\psi_1}) \in \mathcal{C}_\psi$ and $(\omega_{\psi_2}, \omega_{\psi_1}, \omega_1) \in \mathcal{C}_\psi$ *implies* $(\omega_{\psi_2}, \omega_2, \omega_1) \in \mathcal{C}_\psi$

   In total we can extend $\mathcal{C}'_\psi$ to a complete cyclic order: As we see the result of the revision with μ_2 is still $\mathrm{Mod}(\psi \circ \mu_2) = \{\omega_2, \omega\}$, however for μ_1 the result is $\mathrm{Mod}(\psi \circ \mu_1) = \{\omega_1\}$ and therefore $\mathrm{Mod}((\psi \circ \mu_1) \wedge (\psi \circ \mu_2)) = \emptyset$ implies $\psi \circ (\mu_1 \vee \mu_2)$.

4. *At first glance the case $\mathcal{C}'_\psi = \mathcal{C}_\psi \cup \{(\omega_{\psi_1}, \omega, \omega_2), (\omega_{\psi_2}, \omega, \omega_1)\}$ could lead to a revision that violates (R8). However we cannot extend $\mathcal{C}'_\psi$ to a cyclic order because:*

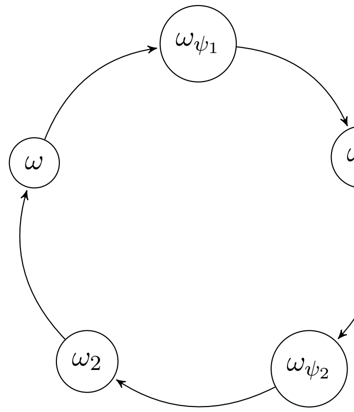

Figure 7: Cyclic order in case 2 of example 7

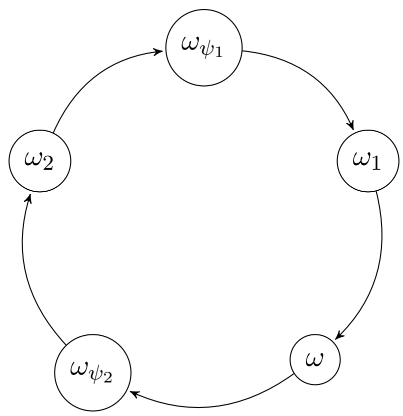

Figure 8: Cyclic order in case 3 of example 7

a) $(\omega_{\psi_1}, \omega_1, \omega) \in \mathcal{C}_\psi$ and $(\omega_{\psi_1}, \omega, \omega_2) \in \mathcal{C}_\psi$ *implies* $(\omega, \omega_2, \omega_1) \in \mathcal{C}_\psi$

b) $(\omega_{\psi_2}, \omega_2, \omega) \in \mathcal{C}_\psi$ and $(\omega_{\psi_2}, \omega, \omega_1) \in \mathcal{C}_\psi$ *implies* $(\omega, \omega_1, \omega_2) \in \mathcal{C}_\psi$

and this violates asymmetry.

We see that the strong faithful cyclic assignment fixes example 6. Before we show that it also ensures the compliance with postulate (R8) we reformulate part of point 3 of definition 11:

Lemma 13. *Let $\mathcal{C}_\psi$ be a cyclic order on Ω and $(\omega_{\psi_1}, \omega_1, \omega_2) \in \mathcal{C}_\psi$ then for $\omega_{\psi_2} \in \Omega$ and $\omega_{\psi_2} \notin \{\omega_{\psi_1}, \omega_1, \omega_2\}$:*

$(\omega_{\psi_2}, \omega_2, \omega_1) \in \mathcal{C}_\psi$ *is equivalent to* $(\omega_{\psi_1}, \omega_1, \omega_{\psi_2}) \in \mathcal{C}_\psi$ *and* $(\omega_{\psi_1}, \omega_{\psi_2}, \omega_2) \in \mathcal{C}_\psi$

Proof. We show both directions:
'$\Rightarrow$'
Let $(\omega_{\psi_1}, \omega_1, \omega_2) \in \mathcal{C}_\psi$, $\omega_{\psi_2} \notin \{\omega_{\psi_1}, \omega_1, \omega_2\}$ and $(\omega_{\psi_2}, \omega_2, \omega_1) \in \mathcal{C}_\psi$, because $\mathcal{C}_\psi$ is cyclic, it follows that $\{(\omega_1, \omega_2, \omega_{\psi_1}), (\omega_1, \omega_{\psi_2}, \omega_2), (\omega_2, \omega_{\psi_1}, \omega_1), (\omega_2, \omega_1, \omega_{\psi_2})\} \subseteq \mathcal{C}_\psi$ and since $\mathcal{C}_\psi$ is transitive $\{(\omega_1, \omega_{\psi_2}, \omega_{\psi_1}), (\omega_2, \omega_{\psi_1}, \omega_2)\} \subseteq \mathcal{C}_\psi$. We use the argument that $\mathcal{C}_\psi$ is cyclic again and receive $(\omega_{\psi_1}, \omega_1, \omega_{\psi_2}) \in \mathcal{C}_\psi$ and $(\omega_{\psi_1}, \omega_{\psi_2}, \omega_2) \in \mathcal{C}_\psi$.
'$\Leftarrow$'
Let $(\omega_{\psi_1}, \omega_1, \omega_{\psi_2}) \in \mathcal{C}_\psi$ and $(\omega_{\psi_1}, \omega_{\psi_2}, \omega_2) \in \mathcal{C}_\psi$. Since $\mathcal{C}_\psi$ is cyclic it follows that $(\omega_{\psi_2}, \omega_{\psi_1}, \omega_1) \in \mathcal{C}_\psi$ and $(\omega_{\psi_2}, \omega_2, \omega_{\psi_1}) \in \mathcal{C}_\psi$. Because $\mathcal{C}_\psi$ is transitive $(\omega_{\psi_2}, \omega_2, \omega_1) \in \mathcal{C}_\psi$ must hold. □

The strong cyclic faithful assignment is sufficient to satisfy (R8):

Theorem 14. *Let $\psi \in \mathcal{L}$ be a knowledge base, $\mu \in \mathcal{L}$ a piece of new information, $\mathcal{C}_\psi$ a strong faithful cyclic order on Ω for $\psi \in \mathcal{L}$ and $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$ if $\mathrm{Mod}(\psi) \neq \emptyset$ else $\emptyset \neq \mathcal{M}_\psi \subseteq \Omega$. The belief-change operator*

$$\mathrm{Mod}(\psi \circ \mu) = \begin{cases} \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) & \text{if } \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \text{else} \end{cases}$$

satisfies postulate (R8).

Proof. We first consider the trivial cases:
If $\mathrm{Mod}(\psi \circ \mu_1) \cap \mathrm{Mod}(\psi \circ \mu_2) = \emptyset$ then $\emptyset \subseteq \mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2))$ i.e. $(\psi \circ \mu_1) \wedge (\psi \circ \mu_2)$ implies $\psi \circ (\mu_1 \vee \mu_2)$.
If $\omega \in \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_1) \cap \mathrm{Mod}(\mu_2)$ then $\omega \in \mathrm{Mod}(\psi) \cap (\mathrm{Mod}(\mu_1) \cup \mathrm{Mod}(\mu_2))$.

For the nontrivial case:
Let $\omega \in \Omega$ such that $\omega \in \mathrm{Mod}(\psi \circ \mu_1)$ and $\omega \in \mathrm{Mod}(\psi \circ \mu_2)$. Then there exists $\emptyset \neq \mathcal{M}_{\psi,\omega}^{(1,2)} \subseteq \mathcal{M}_\psi$ such that for all $\omega_\psi \in \mathcal{M}_{\psi,\omega}^{(1,2)}$ and every $\omega_1 \in \mathrm{Mod}(\mu_1)$ and every $\omega_2 \in \mathrm{Mod}(\mu_2)$: $(\omega_\psi, \omega_1, \omega) \notin \mathcal{C}_\psi$ and $(\omega_\psi, \omega_2, \omega) \notin \mathcal{C}_\psi$. We prove this by contradiction:

We assume that $\mathcal{M}_{\psi,\omega}^{(1,2)} = \emptyset$. Because ω is close to $\mathcal{M}_\psi$ with regards to both $\mathrm{Mod}(\mu_1)$ and $\mathrm{Mod}(\mu_2)$ there exist $\omega_\psi^1, \omega_\psi^2 \in \mathcal{M}_\psi$ *and* $\omega_1 \in Mod(\mu_1)$, $\omega_2 \in$

$Mod(\mu_2)$ such that $(\omega_\psi^1, \omega_2, \omega) \in \mathcal{C}_\psi$ and $(\omega_\psi^2, \omega_1, \omega) \in \mathcal{C}_\psi$ but for all $\tilde{\omega}_1 \in \mathrm{Mod}(\mu_1)$, $\tilde{\omega}_2 \in \mathrm{Mod}(\mu_2):$ $(\omega_\psi^1, \tilde{\omega}_1, \omega) \notin \mathcal{C}_\psi$ and $(\omega_\psi^2, \tilde{\omega}_2, \omega) \notin \mathcal{C}_\psi$. Because of condition 3. of strong faithful assignments, both $(\omega_\psi^1, \omega_2, \omega)$ and $(\omega_\psi^2, \omega_1, \omega)$ have implications on ω_ψ^1 and ω_ψ^2:

1. $(\omega_\psi^1, \omega_2, \omega) \in \mathcal{C}_\psi$ implies either $(\omega_\psi^2, \omega_2, \omega) \in \mathcal{C}_\psi$, or $\{(\omega_\psi^1, \omega_2, \omega_\psi^2), (\omega_\psi^1, \omega_\psi^2, \omega)\} \subseteq \mathcal{C}_\psi$. Because $\omega_2 \in \mathrm{Mod}(\mu_2)$ it follows that $(\omega_\psi^2, \omega_2, \omega) \notin \mathcal{C}_\psi$ and therefore $\{(\omega_\psi^1, \omega_2, \omega_\psi^2), (\omega_\psi^1, \omega_\psi^2, \omega)\} \subseteq \mathcal{C}_\psi$.

2. $(\omega_\psi^2, \omega_1, \omega) \in \mathcal{C}_\psi$ implies either $(\omega_\psi^1, \omega_1, \omega) \in \mathcal{C}_\psi$, or $\{(\omega_\psi^2, \omega_1, \omega_\psi^1), (\omega_\psi^2, \omega_\psi^1, \omega)\} \subseteq \mathcal{C}_\psi$. Because $\omega_1 \in \mathrm{Mod}(\mu_1)$ it follows that $(\omega_\psi^1, \omega_1, \omega) \notin \mathcal{C}_\psi$ and therefore $\{(\omega_\psi^2, \omega_1, \omega_\psi^1), (\omega_\psi^2, \omega_\psi^1, \omega)\} \subseteq \mathcal{C}_\psi$.

However $\{(\omega_\psi^1, \omega_\psi^2, \omega), (\omega_\psi^2, \omega_\psi^1, \omega)\} \subseteq \mathcal{C}_\psi$ violates asymmetry. Therefore $\mathcal{M}_{\psi,\omega}^{(1,2)}$ is not empty.

Hence such a $\omega_\psi \in \mathcal{M}_{\psi,\omega}^{(1,2)}$ exists. Therefore ω is close to $\mathcal{M}_\psi$ with respect to $\mathrm{Mod}(\mu_1 \vee \mu_2)$ and $\omega \in \mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2))$. That means $\psi \circ \mu_1 \wedge \psi \circ \mu_2$ implies $\psi \circ (\mu_1 \vee \mu_2)$. $\square$

With this we can formulate a representation theorem for cyclic orders similar to Katsuno and Mendelzon:

Theorem 15. *A revision operator $\circ$ satisfies conditions (R1) to (R5), (R7) and (R8) if and only if there exists a strong faithful cyclic assignment that maps each knowledge base ψ to a cyclic order $\mathcal{C}_\psi$ such that*

$$\mathrm{Mod}(\psi \circ \mu) = \begin{cases} \mathcal{M}_\psi \cap \mathrm{Mod}(\mu) & \textit{if } \mathcal{M}_\psi \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \textit{else} \end{cases}$$

Proof. (Only-if) Let $\circ$ be a revision operator satisfying postulates (R1) to (R5), (R7) and (R8). We define a relation $\leqslant_\psi$ for each ψ on Ω such that for any worlds $\omega, \tilde{\omega} \in \Omega:$ $\omega \leqslant_\psi \tilde{\omega}$ if and only if either $\omega \in \mathrm{Mod}(\psi)$ or $\mathrm{Mod}(\psi \circ \mathrm{form}(\omega, \tilde{\omega})) = \{\omega\}$. This is a partial pre-order [KM91] and with the strict part $<_\psi$, that behaves like the strict part of a partial order [KM91], we can define a cyclic order $\mathcal{C}_\psi = \{(\omega_1, \omega_2, \omega_3) | (\omega_1 <_\psi \omega_2 <_\psi \omega_3) \vee (\omega_2 <_\psi \omega_3 <_\psi \omega_1) \vee (\omega_3 <_\psi \omega_1 <_\psi \omega_2)\}$ [Nov82]. If ψ is satisfiable then $\mathrm{Mod}(\psi) \neq \emptyset$ and we set $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$, else $\mathcal{M}_\psi = \mathrm{Min}(\Omega, \leqslant_\psi)$.

This assignment is a strong faithful cyclic assignment because

1. $\mathcal{C}_\psi$ is a cyclic order,

2. $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$ if $\mathrm{Mod}(\psi) \neq \emptyset$ and $\mathcal{M}_\psi \neq \emptyset$

3. If $\omega_\psi \in \mathcal{M}_\psi$ and $(\omega_\psi, \omega_1, \omega_2) \in \mathcal{C}_\psi$ then $\omega_\psi <_\psi \omega_1$ and $\omega_1 <_\psi \omega_2$. For all $\tilde{\omega_\psi} \in \mathcal{M}_\psi :$ $\tilde{\omega}_\psi \in \mathrm{Min}(\Omega, \leqslant_\psi)$ and because $<_\psi$ is transitive $\tilde{\omega}_\psi <_\psi \omega_1 <_\psi \omega_2$ which implies $(\tilde{\omega}_\psi, \omega_1, \omega_2) \in \mathcal{C}_\psi$.

Next we show

$$\mathrm{Mod}(\psi \circ \mu) \supseteq \begin{cases} \mathcal{M}_\psi \cap \mathrm{Mod}(\mu) & \text{if } \mathcal{M}_\psi \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \text{else} \end{cases}$$

If $\mathcal{M}_\psi \cap \mathrm{Mod}(\mu) \neq \emptyset$ then $\mathcal{M}_\psi \cap \mathrm{Mod}(\mu) \subseteq \mathrm{Mod}(\psi \circ \mu)$ because of (R2).

If $\mathcal{M}_\psi \cap \mathrm{Mod}(\mu) = \emptyset$ then let $\omega \in \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi)$ and $\mathrm{Mod}(\mu) = \{\omega_1, ..., \omega_\mathrm{n}\}$.

Because $\omega \in \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi)$ there exists some $\omega_\psi \in \mathcal{M}_\psi$ such that for all $\omega_\mathrm{i} \in \mathrm{Mod}(\mu) : (\omega_\psi, \omega_\mathrm{i}, \omega) \notin \mathcal{C}_\psi$. Because $\mathrm{Mod}(\mu) \cap \mathcal{M}_\psi = \emptyset$ for all $\omega_\mathrm{i} \in \mathrm{Mod}(\mu) : \omega_\psi <_\psi \omega_\mathrm{i}$ and therefore $\omega_\mathrm{i} \not<_\psi \omega$, otherwise it would follow that if $\omega_\psi <_\psi \omega_\mathrm{i} <_\psi \omega$ then $(\omega_\psi, \omega_\mathrm{i}, \omega) \in \mathcal{C}_\psi$. As a consequence for all $\omega_\mathrm{i} \in \mathrm{Mod}(\mu)$:

$\omega \in \mathrm{Mod}(\psi \circ \mathrm{form}(\omega, \omega_\mathrm{i}))$

As a result

$\omega \in \mathrm{Mod}((\psi \circ \mathrm{form}(\omega, \omega_1) \wedge ... \wedge (\psi \circ \mathrm{form}(\omega, \omega_\mathrm{n})))$.

Because of (R8), this implies that ω is a model of

$\psi \circ (\mathrm{form}(\omega, \omega_1) \vee ... \vee \mathrm{form}(\omega, \omega_\mathrm{n}))$

i.e. $\omega \in \mathrm{Mod}(\psi \circ \mu)$ (see [KM91]).

(if) We know that a change operator defined on strong faithful cyclic orders satisfies postulates (R1) to (R5), (R7) and (R8). Therefore we only have to show the other inclusion

$$\mathrm{Mod}(\psi \circ \mu) \subseteq \begin{cases} \mathcal{M}_\psi \cap \mathrm{Mod}(\mu) & \text{if } \mathcal{M}_\psi \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \text{else} \end{cases}$$

If $\mathcal{M}_\psi \cap \mathrm{Mod}(\mu) \neq \emptyset$ then $\psi \wedge \mu$ is satisfiable and because of (R2) it follows that $\psi \circ \mu \equiv \psi \wedge \mu$, i.e. $\mathrm{Mod}(\psi \circ \mu) = \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) = \mathcal{M}_\psi \cap \mathrm{Mod}(\mu)$.

If $\mathcal{M}_\psi \cap \mathrm{Mod}(\mu) = \emptyset$, we know that $\mathrm{Mod}(\psi \circ \mu) = \mathrm{Min}(\mathrm{Mod}(\mu), \leqslant_\psi)$ [KM91]. Let $\omega \in \mathrm{Mod}(\psi \circ \mu)$ then for $\omega_\psi \in \mathcal{M}_\psi$ and for all $\omega_\mathrm{i} \in \mathrm{Mod}(\mu) : \omega_\psi <_\psi \omega_\mathrm{i} \not<_\psi \omega$. Therefore for all $\omega_\mathrm{i} \in \mathrm{Mod}(\mu) : (\omega_\psi, \omega_\mathrm{i}, \omega) \notin \mathcal{C}_\psi$ i.e. ω is close to $\mathrm{Mod}(\mu)$ in $\mathcal{C}_\psi$ with respect to $\mathrm{Mod}(\mu)$. $\square$

4 Discussion

Our stricter version of a faithful cyclic assignment has made it possible to only consider a class of cyclic orders that satisfy (R8). As previously stated there is another possible way to formulate a representation theorem for cyclic orders. For this we need to find additional supplemental postulates. In order to find such postulates, we have to investigate the cyclic orders that do not satisfy (R8).

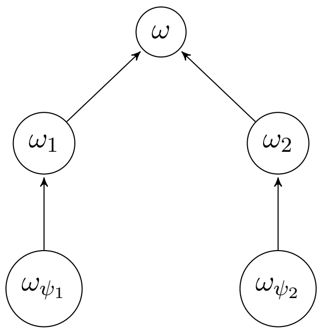

Figure 9: Part of cyclic order 1 in example 8

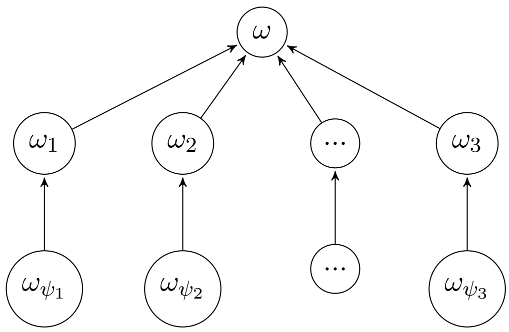

Figure 10: Part of cyclic order 2 in example 8

4.1 Blocking Sets

In the following example we investigate a few cyclic orders that do not satisfy (R8) in order to find a common pattern.

Example 8. *In our counterexample $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_2}, \omega_2, \omega)\}$ for (R8), the world ω is not close to $\mathcal{M}_\psi$ only if ω_1 and ω_2 are in the same set. If we interpret this as a graph, these two worlds 'block' the paths from $\mathcal{M}_\psi = \{\omega_{\psi_1}, \omega_{\psi_2}\}$ to ω, see figure 8. We can generalise this example $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_2}, \omega_2, \omega), \dots, (\omega_{\psi_n}, \omega_n, \omega)\}$, see figure 8. As we can see here $\mathrm{Mod}(\psi \circ \mathrm{form}(\omega_1, \omega_2, \omega)) = \{\omega_1, \omega_2, \omega\}$ and for any $\mathcal{N} \subsetneq \{\omega_1, \omega_2, ..., \omega_n\}$ it follows that $\mathrm{Mod}(\psi \circ \mathrm{form}(\mathcal{N}, \omega)) = \mathcal{N} \cup \{\omega\}$ however $\mathrm{Mod}(\psi \circ \mathrm{form}(\omega_1, \omega_2, ..., \omega_n, \omega)) = \{\omega_1, \omega_2, ..., \omega_n\}$.*

Another cyclic order that does not satisfy (R8) is $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega_a), (\omega_{\psi_2}, \omega_2, \omega_a), (\omega_{\psi_1}, \omega_1, \omega_b), (\omega_{\psi_2}, \omega_2, \omega_b)\}$, see figure 8. However we can reduce this case to two instances of our initial simple case.. A related example is $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_2}, \omega_2, \omega), (\omega_{\psi_2}, \omega_3, \omega)\}$, see figure 8. Here it is important to note that $\mu_1 \equiv \mathrm{form}(\omega_1, \omega)$ and $\mu_2 \equiv \mathrm{form}(\omega_2, \omega_3, \omega)$ leads to $\mathrm{Mod}((\psi \circ \mu_1) \wedge (\psi \circ \mu_2)) = \{\omega\}$ and $\mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2)) = \{\omega_1, \omega_2, \omega_3\}$. However

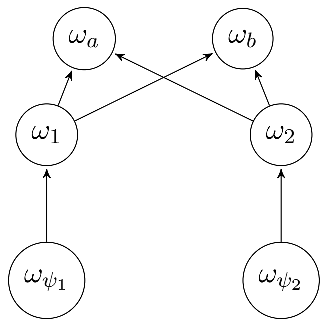

Figure 11: Part of cyclic order 3 in example 8

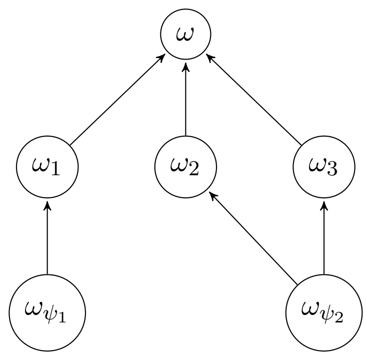

Figure 12: Part of cyclic order 4 in example 8

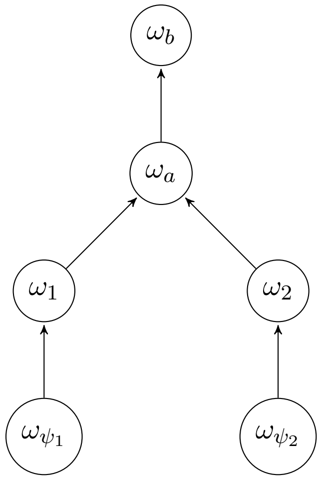

Figure 13: Part of cyclic order 5 in example 8

both $\mathcal{N}_1 = \{\omega_1, \omega_2\}$ *and* $\mathcal{N}_2 = \{\omega_1, \omega_3\}$ *are already sufficient to 'block'* ω. *Furthermore* $\psi \circ \mathrm{form}(\omega_2, \omega_3, \omega) \equiv \mathrm{form}(\omega_2, \omega_3, \omega)$, *i.e. the set* $\mathcal{N}_3 = \{\omega_2, \omega_3\}$ *is not enough to 'block'* ω.

A possible case where there is a preference in the part that contradicts (R8) is $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega_\mathrm{a}), (\omega_{\psi_2}, \omega_2, \omega_\mathrm{a}), (\omega_{\psi_1}, \omega_1, \omega_\mathrm{b}), (\omega_{\psi_2}, \omega_2, \omega_\mathrm{b}), (\omega_{\psi_1}, \omega_\mathrm{a}, \omega_\mathrm{b}), (\omega_{\psi_2}, \omega_\mathrm{a}, \omega_\mathrm{b})\}$, *see figure 8. We see that the condition that contradicts (R8) is transitive in cyclic orders, because cyclic orders are transitive and therefore additional postulates should express this.*

From what we gathered in the previous example, we have found a condition that contradicts (R8). We give this condition a name:

Definition 12. *Let* $\mathcal{N} \subseteq \Omega$, $\omega \in \Omega$, $\psi \in \mathcal{L}$ *and* $\circ$ *a belief-change operator that satisfies the postulates (R1) to (R5) and (R7). We call* $\mathcal{N}$ *an* blocking set *for* ω *if for all* $\mathcal{N}' \subsetneq \mathcal{N} :\ \mathrm{Mod}(\psi \circ \mathrm{form}(\mathcal{N}', \omega)) = \mathcal{N}' \cup \{\omega\}$ *and* $\mathrm{Mod}(\psi \circ \mathrm{form}(\mathcal{N}, \omega)) = \mathcal{N}$.

When we look at the proof of theorem 15 and also the proof of the representation theorem in the work of Katsuno and Mendelzon [KM91], postulate (R8) is only used for the proof of the inclusions $\mathrm{Min}(\mathrm{Mod}(\mu), \leqslant_\psi)) \subseteq \mathrm{Mod}(\psi \circ \mu)$ and $\mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) \subseteq \mathrm{Mod}(\psi \circ \mu)$. However we need to consider that we cannot use the proposed construction of a faithful cyclic order for a belief-change operator $\circ$ and $\psi \in \mathcal{L}$, because:

Example 9. *For our standard example* $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_2}, \omega_2, \omega)\}$ *with* $\mathrm{Mod}(\psi) = \{\omega_{\psi_1}, \omega_{\psi_2}\}$ *we consider the belief-change operator* $\circ$ *defined by* $\mathcal{C}_\psi$*:*

$$\mathrm{Mod}(\psi \circ \mu) = \begin{cases} \mathcal{M}_\psi \cap \mathrm{Mod}(\mu) & \text{if } \mathcal{M}_\psi \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \text{else} \end{cases}$$

and construct the corresponding binary partial order $<_\psi$:

1. $\omega_{\psi_1} <_\psi \omega_1$ because $\mathrm{Mod}(\psi \circ \mathrm{form}(\omega_{\psi_1}, \omega_1)) = \{\omega_{\psi_1}\}$,
2. $\omega_{\psi_1} <_\psi \omega_2$ because $\mathrm{Mod}(\psi \circ \mathrm{form}(\omega_{\psi_1}, \omega_2)) = \{\omega_{\psi_1}\}$,
3. $\omega_{\psi_1} <_\psi \omega$ because $\mathrm{Mod}(\psi \circ \mathrm{form}(\omega_{\psi_1}, \omega)) = \{\omega_{\psi_1}\}$,
4. $\omega_{\psi_2} <_\psi \omega_1$ because $\mathrm{Mod}(\psi \circ \mathrm{form}(\omega_{\psi_2}, \omega_1)) = \{\omega_{\psi_2}\}$,
5. $\omega_{\psi_2} <_\psi \omega_2$ because $\mathrm{Mod}(\psi \circ \mathrm{form}(\omega_{\psi_2}, \omega_2)) = \{\omega_{\psi_2}\}$,
6. $\omega_{\psi_2} <_\psi \omega$ because $\mathrm{Mod}(\psi \circ \mathrm{form}(\omega_{\psi_2}, \omega)) = \{\omega_{\psi_2}\}$.

The cyclic order we receive from the binary order $<_\psi$ is $\mathcal{C}'_\psi = \emptyset$ and accordingly $\mathrm{closest}(\{\omega_1, \omega_2, \omega\}, \mathcal{M}_\psi, \mathcal{C}'_\psi) = \{\omega_1, \omega_2, \omega\} \neq \{\omega_1, \omega_2\} = \mathrm{closest}(\{\omega_1, \omega_2, \omega\}, \mathcal{M}_\psi, \mathcal{C}_\psi)$.

Thus the construction used for the proof of theorem 15 cannot be used in a possible representation theorem for weak faithful cyclic orders.

Before we study blocking sets in detail we formulate a postulate (C1) that extends the transitivity postulate (Rt) of Katsuno and Mendelzon [KM91]. We call the following three version (C1a), (C1b) and (C1c) together (C1):

(C1a) If $\psi \circ (\mu \vee \phi \vee \alpha) \wedge \mu$ is satisfiable then $\psi \circ (\mu \vee \phi) \wedge \mu$ is satisfiable.

(C1b) If $\psi \circ (\mu \vee \phi) \wedge \mu$ is not satisfiable then $\psi \circ (\mu \vee \phi \vee \alpha) \wedge \mu$ is not satisfiable.

(C1c) If $\psi \circ (\mu \vee \phi \vee \alpha) \equiv \mu$ then $\psi \circ (\mu \vee \phi) \equiv \mu$.

Theorem 16. *Let $\psi \in \mathcal{L}$ be a knowledge base, $\mu \in \mathcal{L}$ a piece of new information, $\mathcal{C}_\psi$ a faithful cyclic order on Ω and $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$ if $\mathrm{Mod}(\psi) \neq \emptyset$ else $\emptyset \neq \mathcal{M}_\psi \subseteq \Omega$. The belief-change operator*

$$\mathrm{Mod}(\psi \circ \mu) = \begin{cases} \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) & \textit{if } \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \textit{else} \end{cases}$$

satisfies (C1a), (C1b) and (C1c).

Proof. (C1a) If $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu \vee \phi\alpha) \neq \emptyset$ then because $\psi \circ (\mu \vee \phi \vee \alpha) \wedge \mu$ is satisfiable it follows that $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$ and therefore $\psi \circ (\mu \vee \phi) \wedge \mu$ is satisfiable. If $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu \vee \phi\alpha) = \emptyset$ then because $\psi \circ (\mu \vee \phi \vee \alpha) \wedge \mu$ is satisfiable there exists $\omega \in \mathrm{Mod}(\mu)$ such that for some $\omega_\psi \in \mathcal{M}_\psi$ no $\tilde{\omega} \in \mathrm{Mod}(\mu \vee \phi \vee \alpha)$ satisfies $(\omega_\psi, \tilde{\omega}, \omega) \in \mathcal{C}_\psi$. Because $\mathrm{Mod}(\mu \vee \phi) \subseteq \mathrm{Mod}(\mu \vee \phi \vee \alpha)$ there is no $\tilde{\omega} \in \mathrm{Mod}(\mu \vee \phi)$ that satisfies $(\omega_\psi, \tilde{\omega}, \omega) \in \mathcal{C}_\psi$ and therefore ω is close to $\mathcal{M}_\psi$ in $\mathcal{C}_\psi$ with respect to

$\mathrm{Mod}(\mu \vee \phi)$.
(C1b) Towards contradiction we assume that $\psi \circ (\mu \vee \phi) \wedge \mu$ is not satisfiable, but $\psi \circ (\mu \vee \phi \vee \alpha) \wedge \mu$ is satisfiable. Because of (C1a) it follows that $\psi \circ (\mu \vee \phi) \wedge \mu$ is satisfiable, which is a contradiction.
(C1c) With the same argument used for (C1a) every $\omega \in \mathrm{Mod}(\mu)$ is close to $\mathcal{M}_\psi$ with respect to $\mathrm{Mod}(\mu \vee \phi \vee \alpha)$ in $\mathcal{C}_\psi$ and therefore also with respect to $\mathrm{Mod}(\mu \vee \phi)$. □

The following postulate ensures that blocking sets have transitive properties: (C2) If

1. $\psi \circ (\mu \vee \phi) \wedge \mu$ is satisfiable,
2. $\psi \circ (\mu \vee \alpha) \wedge \mu$ is satisfiable,
3. $\psi \circ (\mu \vee \phi \vee \alpha) \wedge \mu$ is not satisfiable and
4. $\psi \circ (\alpha \vee \beta) \equiv \beta$

then $\psi \circ (\mu \vee \phi \vee \beta) \wedge \mu$ is not satisfiable.

Theorem 17. *Let $\psi \in \mathcal{L}$ be a knowledge base, $\mu \in \mathcal{L}$ a piece of new information, $\mathcal{C}_\psi$ a faithful cyclic order on Ω and $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$ if $\mathrm{Mod}(\psi) \neq \emptyset$ else $\emptyset \neq \mathcal{M}_\psi \subseteq \Omega$. The belief-change operator*

$$\mathrm{Mod}(\psi \circ \mu) = \begin{cases} \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) & \textit{if } \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{closest}(\mathrm{Mod}(\mu), \mathcal{M}_\psi, \mathcal{C}_\psi) & \textit{else} \end{cases}$$

satisfies (C2).

Proof. (C2) Towards contradiction we assume that $\psi \circ (\mu \vee \phi \vee \beta) \wedge \mu$ is satisfiable. Then there exists some $\omega \in \mathrm{Mod}(\mu)$ and some $\omega_\psi \in \mathcal{M}_\psi$ such that for all $\tilde{\omega} \in \mathrm{Mod}(\mu \vee \phi \vee \beta) : (\omega_\psi, \tilde{\omega}, \omega) \notin \mathcal{C}_\psi$. Because $\psi \circ (\mu \vee \phi \vee \alpha) \wedge \mu$ is not satisfiable and $\psi \circ (\mu \vee \phi) \wedge \mu$ is satisfiable it follows that there exists some $\omega_\alpha \in \mathrm{Mod}(\alpha)$ such that $(\omega_\psi, \omega_\alpha, \omega) \in \mathcal{C}_\psi$. Furthermore because of $\psi \circ (\alpha \vee \beta) \equiv \beta$ it follows that there exists some $\omega_\beta \in \mathrm{Mod}(\beta)$ such that $(\omega_\psi, \omega_\beta, \omega_\alpha) \in \mathcal{C}_\psi$ and with transitivity $(\omega_\psi, \omega_\beta, \omega) \in \mathcal{C}_\psi$. That is a contradiction and therefore $\psi \circ (\mu \vee \phi \vee \beta)$ is not satisfiable. □

4.2 Properties of blocking sets

A significant property of blocking sets in weak faithful cyclic orders is the following:

Lemma 18. *Let $|\mathcal{M}_\psi| = \mathrm{n}$, $\mathcal{C}_\psi$ a cyclic order, $\omega \in \Omega$ and $\circ$ the change operator received from $\mathcal{C}_\psi$ then the maximum size of a blocking set $\mathcal{N} \subseteq \Omega$ for ω is n.*

Proof. We proof that for every $\omega_\mathrm{i} \in \mathcal{N}$ there is at least one $\omega_{\psi_\mathrm{j}} \in \mathcal{M}_\psi$ that is exclusive to ω_i in the sense that $(\omega_{\psi_\mathrm{j}}, \omega_\mathrm{i}, \omega) \in \mathcal{C}_\psi$ and for all $\omega_\mathrm{i} \neq \omega_\mathrm{k} \in \mathcal{N} :\ (\omega_{\psi_\mathrm{j}}, \omega_\mathrm{k}, \omega) \notin \mathcal{C}_\psi$.

Let $\mathcal{N}$ be a blocking set for ω then for all $\omega_\mathrm{i} \in \mathcal{N}$, it follows that $\mathrm{Mod}(\psi \circ \mathrm{form}(\mathcal{N} \backslash \{\omega_\mathrm{i}\}, \omega)) = (\mathcal{N} \backslash \{\omega_\mathrm{i}\}) \cup \{\omega\}$. Therefore there exists $\omega_{\psi_\mathrm{j}} \in \mathcal{M}_\psi$ such that for all $\omega_\mathrm{k} \in \mathcal{N} :\ (\omega_{\psi_\mathrm{j}}, \omega_\mathrm{k}, \omega) \notin \mathcal{C}_\psi$. On the other hand $\mathrm{Mod}(\psi \circ \mathrm{form}(\mathcal{N}, \omega)) = \mathcal{N}$ and therefore for ω_{ψ_j} exists some $\omega^* \in \mathcal{N}$ such that $(\omega_{\psi_\mathrm{j}}, \omega^*, \omega) \in \mathcal{C}_\psi$ which implies $\omega^* = \omega_\mathrm{i}$. Therefore for every $\omega_\mathrm{i} \in \mathcal{N}$ there is at least one $\omega_{\psi_\mathrm{j}} \in \mathcal{M}_\psi$ that is exclusive to ω_i. Consequently the maximum size of $\mathcal{N}$ can only be n. □

The implications from this are, that on the one hand, if $|\mathcal{M}_\psi| = 1$ the cyclic orders satisfy (R8) and on the other hand, that there are belief change operators that do not satisfy (R8) and cannot be mapped to a cyclic order.

Example 10. *Let $\mathcal{L}$ have two propositional letters a and b. We consider the following possible worlds:*

$$\omega_1 = \langle 1, 1 \rangle,$$
$$\omega_2 = \langle 1, 0 \rangle,$$
$$\omega_3 = \langle 0, 1 \rangle,$$
$$\omega_4 = \langle 0, 0 \rangle.$$

Let

$$\psi \equiv \mathrm{form}(\omega_1),$$
$$\mu_1 \equiv \mathrm{form}(\omega_2, \omega_4),$$
$$\mu_2 \equiv \mathrm{form}(\omega_3, \omega_4).$$

For two possible worlds ω_1, ω_2 and $\mathrm{a} \in \Sigma$ we denote by $\mathrm{dist_a}(\omega_\mathrm{i}, \omega_\mathrm{j})$ the distance of the two possible worlds $\omega_\mathrm{i},\ \omega_\mathrm{j}$ in the propositional variable a. It is 0 if both have the same value and 1 if their value is different, i.e. $|\omega_\mathrm{i}^\mathrm{a} - \omega_\mathrm{j}^\mathrm{a}|$, where ω^a means the value of ω in a. We expand this and define a minimal set with respect to a for sets of propositional worlds:

$$\mathrm{min_a}\ \mathrm{dist_a}(\omega_\mathrm{i}, \mathcal{N}) = \{\omega \in \mathcal{N} | \forall \omega_\mathrm{j} \in \mathcal{N} : \mathrm{dist_a}(\omega_\mathrm{i}, \omega_\mathrm{j}) \not< \mathrm{dist_a}(\omega_\mathrm{i}, \omega)\}$$

In other words the set $\mathrm{min_a}\ \mathrm{dist_a}(\omega_\mathrm{i}, \mathcal{N}) \subseteq \mathcal{N}$ is either equal to $\mathcal{N} \subseteq \Omega$ if no $\omega \in \mathcal{N}$ is equal to ω_i regarding the propositional letter a, or the subset of $\mathcal{N}$ of possible worlds that have the same value as ω_i regarding a. When we expand it on all propositional letters:

$$\bigcup_{\mathrm{a} \in \Sigma} \mathrm{min_a}\ \mathrm{dist_a}(\omega_\mathrm{i}, \mathcal{N})$$

it is clear that this set is only then not equal to $\mathcal{N}$ if there is one $\omega \in \mathcal{N}$ such that $\prod_{\mathrm{a} \in \Sigma} \mathrm{dist_a}(\omega_\mathrm{i}, \omega) = 1$ and for all $\mathrm{a} \in \Sigma$ exists $\omega' \in \mathcal{N}$ such that $\mathrm{dist_a}(\omega_\mathrm{i}, \omega') = 0$. For $\omega_1, \omega_2, \omega_3$ and ω_4:

$$\mathrm{dist_a}(\omega_1, \omega_2) = 0,\ \mathrm{dist_b}(\omega_1, \omega_2) = 1,$$
$$\mathrm{dist_a}(\omega_1, \omega_3) = 1,\ \mathrm{dist_b}(\omega_1, \omega_3) = 0,$$

$$\mathrm{dist_a}(\omega_1, \omega_4) = 1,\ \mathrm{dist_b}(\omega_1, \omega_4) = 1.$$

If we would simply define a belief change operator as

$$\mathrm{Mod}(\psi \circ_{\mathrm{h}} \mu) = \bigcup_{\omega_\psi \in \mathrm{Mod}(\psi)} \bigcup_{\mathrm{a} \in \Sigma} \min_{\mathrm{a}} \mathrm{dist_a}(\omega_\psi, \mathrm{Mod}(\mu)),$$

it would not satsify (R2) and (R3). For (R2) we simply define

$$\phi \equiv \mathrm{form}(\omega_1, \omega_2)$$

and see that $\omega_2 \in \min_{\mathrm{a}}\mathrm{dist_a}(\omega_1, \mathrm{Mod}(\mu)) \subseteq \bigcup_{\omega_\psi \in \mathrm{Mod}(\psi)} \bigcup_{\mathrm{a}\in\Sigma} \min_{\mathrm{a}} \mathrm{dist_a}(\omega_\psi, \mathrm{Mod}(\mu))$ *because* $\mathrm{dist_a}(\omega_1, \omega_2) = 0$*. Therefore* $\omega_2 \in \mathrm{Mod}(\psi \circ_{\mathrm{h}} \mu)$ *even though* $\psi \wedge \phi$ *is satisfiable and therefore (R2) is not satisfied. We can prevent this by using the same method we used for cyclic orders.*

For (R3) we need to consider the case $\psi' \equiv \bot$*. We could define* $\mathrm{dist_a}(\emptyset, \omega) = 1$ *and* $\min_{\mathrm{a}}\mathrm{dist_a}(\emptyset, \mathcal{N}) = \mathcal{N}$*. Alternatively we can simply demand* $\mathrm{Mod}(\psi \circ \mu) = \mathrm{Mod}(\mu)$ *if* $\mathrm{Mod}(\psi) = \emptyset$*.*

With this in mind we define a belief change operator $\circ_B$ *as*

$$\mathrm{Mod}(\psi \circ_{\mathrm{B}} \mu) = \begin{cases} \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) & \textit{if } \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset \\ \mathrm{Mod}(\mu) & \textit{if } \mathrm{Mod}(\psi) = \emptyset \\ \displaystyle\bigcup_{\omega_\psi \in \mathrm{Mod}(\psi)} \bigcup_{\mathrm{a}\in\Sigma} \min_{\mathrm{a}} \mathrm{dist_a}(\omega_\psi, \mathrm{Mod}(\mu)) & \textit{else} \end{cases}$$

Before we proof that $\circ_h$ *satisfies postulates (R1) to (R5) and (R7) we want to establish, that this revision is the 'trivial' revision if* $|\mathrm{Mod}(\psi)| \neq 1$*. Suppose* $|\mathrm{Mod}(\psi)| > 1$ *then there are* $\omega_{\psi_1} \in \mathrm{Mod}(\psi)$ *and* $\omega_{\psi_2} \in \mathrm{Mod}(\psi)$ *such that* $\omega_{\psi_1} \neq \omega_{\psi_2}$*. That means that there is some* $\mathrm{a} \in \Sigma$ *such that* $\mathrm{dist_a}(\omega_{\psi_1}, \omega_{\psi_2}) = 1$*. As a result for all* $\alpha \in \mathcal{L}$ *with* $\mathrm{Mod}(\alpha) \neq \emptyset$ *it follows that* $\bigcup_{\omega_\psi \in \mathrm{Mod}(\psi)} \bigcup_{\mathrm{a}\in\Sigma} \min_{\mathrm{a}} \mathrm{dist_a}(\omega_\psi, \mathrm{Mod}(\mu)) = \mathrm{Mod}(\alpha)$ *because let* $\omega_\alpha \in \mathrm{Mod}(\alpha)$ *than either* $\mathrm{dist_a}(\omega_{\psi_1}, \omega_\alpha) = 0$ *or* $\mathrm{dist_a}(\omega_{\psi_2}, \omega_\alpha) = 0$*. With that* $\omega_\alpha \in (\min_{\mathrm{a}}\mathrm{dist_a}(\omega_{\psi_1}, \omega_\alpha) \cup \min_{\mathrm{a}}\mathrm{dist_a}(\omega_{\psi_2}, \omega_\alpha))$*. Alternatively if* $\mathrm{Mod}(\psi) = \emptyset$ *it follows that* $\psi \wedge \mu$ *is never satisfiable and the revision is always* $\mathrm{Mod}(\mu)$*. That means if* $|\mathrm{Mod}(\psi)| \neq 1$ *we can alternatively define* $\circ_{\mathrm{B}}$ *as*

$$\psi \circ_{\mathrm{B}} \mu = \begin{cases} \psi \wedge \mu & \textit{if } \psi \wedge \mu \textit{ is satisfiable} \\ \mu & \textit{else} \end{cases}$$

This operator famously satisfies the AGM-Postulates. Therefore we need to only consider the case $|\mathrm{Mod}(\psi)| = 1$*.*

(R1) and (R2) are satisfied because of the construction of $\circ_{\mathrm{B}}$*. For (R3) we assume towards contradiction that there are some* $\psi, \mu \in \mathcal{L}$ *such that* $\mathrm{Mod}(\mu) \neq \emptyset$ *and* $\mathrm{Mod}(\psi \circ_{\mathrm{B}} \mu)$*. The cases* $\mathrm{Mod}(\psi) = \emptyset$ *and* $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu) \neq \emptyset$ *are trivial. We know that* $\Sigma \neq \emptyset$ *because* $\mathrm{Mod}(\mu) \neq \emptyset$ *and therefore there is some* $\mathrm{a} \in \Sigma$*. Let* $\omega_\psi \in \mathrm{Mod}(\psi)$*, we consider* $\min_{\mathrm{a}}\mathrm{dist_a}(\omega_\psi, \mathrm{Mod}(\mu))$*. If* $\omega_\mu \in \mathrm{Mod}(\mu)$ *not in* $\min_{\mathrm{a}}\mathrm{dist_a}(\omega_\psi, \mathrm{Mod}(\mu))$ *then there*

is some $\omega^{'} \in \mathrm{Mod}(\mu)$ *such that* $\mathrm{dist_a}(\omega_\psi, \omega^{'}) = 0 < 1 = \mathrm{dist_a} < (\omega_\psi, \omega_\mu)$ *and therefore* $\omega^{'} \in \mathrm{min_a dist_a}(\omega_\psi, \mathrm{Mod}(\mu))$ *and therefore* $\mathrm{Mod}(\psi \circ_\mathrm{B} \mu) \neq \emptyset$ *which is a contradiction.*

(R4) is satisfied because we only consider the models of ψ *and* μ *while the syntax has no impact.*

For (R5), let $\psi, \mu, \phi \in \mathcal{L}$ *and* $\omega \in \mathrm{Mod}((\psi \wedge \mu) \wedge \phi) \neq \emptyset$. *If* $\mathrm{Mod}((\psi \wedge \mu) \wedge \phi) \neq \emptyset$ *then* $(\psi \circ_\mathrm{B} \mu) \wedge \phi)$ *implies* $\psi \circ_\mathrm{B} (\mu \wedge \phi)$. *It follows that there is some* $\mathrm{a} \in \Sigma$ *such that* $\omega \in \mathrm{min_a dist_a}(\omega_\psi, \mathrm{Mod}(\mu))$ *and* $\omega \in \mathrm{Mod}(\phi)$. *When we regard the set* $\mathrm{Mod}(\mu \wedge \phi) = \mathrm{Mod}(\mu) \cap \mathrm{Mod}(\phi)$, *then* $\omega \in \mathrm{min_a dist_a}(\omega_\mathrm{p}\mathrm{si}, \mathrm{Mod}(\mu \wedge \phi))$ *because* $(\mathrm{Mod}(\mu) \cap \mathrm{Mod}(\phi)) \subseteq \mathrm{Mod}(\mu)$ *(otherwise there would be a possible world* $\omega^{'} \in \mathrm{Mod}(\mu \wedge \phi) \subseteq \mathrm{Mod}(\mu)$ *such that* $\mathrm{dist_a}(\omega_\psi, \omega^{'}) = 0 < 1 = \mathrm{dist_a}(\omega_\psi, \omega)$ *which is a contradiction etc.). Therefore* $\omega \in \mathrm{Mod}(\psi \circ_\mathrm{B} (\mu \wedge \phi))$ *and hence* $(\psi \circ_\mathrm{B} \mu) \wedge \phi$ *implies* $\psi \circ_\mathrm{B} (\mu \wedge \phi)$.

For (R7), let $\mu_1, \mu_2, \psi \in \mathcal{L}$. *We already discussed the case* $\mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_1)$. *The case* $\mathrm{Mod}(\psi) = \emptyset$ *is also trivial because* $\mathrm{Mod}(\psi \circ_\mathrm{B} \mu_1) = \mathrm{Mod}(\mu_1) \subseteq \mathrm{Mod}(\psi \circ_\mathrm{B} \mu_2) = \mathrm{Mod}(\mu_2)$ *and* $\mathrm{Mod}(\psi \circ_\mathrm{B} \mu_2) = \mathrm{Mod}(\mu_2) \subseteq \mathrm{Mod}(\psi \circ_\mathrm{B} \mu_1) = \mathrm{Mod}(\mu_1)$ *implies* $\mathrm{Mod}(\psi \circ_\mathrm{B} \mu_1) = \mathrm{Mod}(\mu_1) = \mathrm{Mod}(\mu_2) = \mathrm{Mod}(\psi \circ_\mathrm{B} \mu_2)$.

Let $\mathrm{Mod}(\mu_1) \neq \emptyset = \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\mu_1)$, $\mathrm{Mod}(\psi \circ_\mathrm{B} \mu_1) \subseteq \mathrm{Mod}(\mu_2)$ *and* $\mathrm{Mod}(\psi \circ_\mathrm{B} \mu_2) \subseteq \mathrm{Mod}(\mu_1)$. *Towards contradiction we assume that* $\psi \circ_\mathrm{B} \mu_1 \not\equiv \psi \circ_\mathrm{B} \mu_2$. *Then without loss of generality there exists some* $\omega \in \mathrm{Mod}(\psi \circ_\mathrm{B} \mu_1)$ *such that* $\omega \notin \mathrm{Mod}(\psi \circ_\mathrm{B} \mu_2)$. *Because* $\omega \in \mathrm{Mod}(\psi \circ_\mathrm{B} \mu_1)$ *there exists* $\mathrm{a} \in \Sigma$ *such that* $\omega \in \mathrm{min_a dist_a}(\omega_\psi, \mathrm{Mod}(\mu_1))$, *i.e. for all* $\omega^{'} \in \mathrm{Mod}(\mu) : \mathrm{dist_a}(\omega_\psi, \omega^{'}) \not< \mathrm{dist_a}(\omega_\psi, \omega)$. *Since* $\omega \notin \mathrm{Mod}(\psi \circ_\mathrm{B} \mu_2)$ *for all* $\mathrm{b} \in \Sigma : \omega \notin \mathrm{min_b dist_b}(\omega_\psi, \mathrm{Mod}(\mu_2))$, *i.e. for all* $\mathrm{b} \in \Sigma$ *exists* $\omega* \in \mathrm{Mod}(\mu_2)$ *such that* $\mathrm{dist_b}(\omega_\psi, \omega^*) = 0 < 1 = \mathrm{dist_b}(\omega_\psi, \omega)$. *Let* $\omega_\mathrm{a} \in \mathrm{Mod}(\mu_2)$ *such that* $\mathrm{dist_a}(\omega_\psi, \omega_\mathrm{a}) = 0 < 1 = \mathrm{dist_a}(\omega_\psi, \omega)$ *then* $\omega_\mathrm{a} \in \mathrm{Mod}(\mu_1)$ *because* $\psi \circ_B \mu_2$ *implies* μ_1. *That is a contradiction and therefore* $\circ_\mathrm{B}$ *does satisfy (R7).*

For (R8) we revisit $\omega_1, \omega_2, \omega_3, \omega_3, \psi, \mu_1$ *and* μ_2 *from above. We see that:*

$$\mathrm{min_a dist_a}(\omega_1, \{\omega_2, \omega_4\}) = \{\omega_2\},$$

$$\mathrm{min_a dist_a}(\omega_1, \{\omega_3, \omega_4\}) = \{\omega_3, \omega_4\},$$

$$\mathrm{min_a dist_a}(\omega_1, \{\omega_2, \omega_3, \omega_4\}) = \{\omega_2\},$$

$$\mathrm{mind_b dist_b}(\omega_1, \{\omega_2, \omega_4\}) = \{\omega_2, \omega_4\},$$

$$\mathrm{mind_b dist_b}(\omega_1, \{\omega_3, \omega_4\}) = \{\omega_3\},$$

$$\mathrm{mind_b dist_b}(\omega_1, \{\omega_2, \omega_3, \omega_4\}) = \{\omega_3\}.$$

It follows that $\omega_4 \in \mathrm{Mod}(\psi \circ_\mathrm{B} \mu_1) \cap \mathrm{Mod}(\psi \circ_\mathrm{B} \mu_2)$ *but* $\omega_4 \notin \mathrm{Mod}(\psi \circ_\mathrm{B} (\mu_1 \vee \mu_2)) = \mathrm{min_a dist_a}(\omega_1, \{\omega_2, \omega_3, \omega_4\}) \cup \mathrm{min_b dist_b}(\omega_1, \{\omega_2, \omega_3, \omega_4\})$. *We see that* $\mathcal{N} = \{\omega_2, \omega_3\}$ *is a blocking set for* ω_4 *but because* $|\mathrm{Mod}(\psi)| = 1$ *there is no weak faithful cyclic order that could represent this case.*

The following theorem formalizes that blocking sets are necessary in order to violate (R8):

Theorem 19. *Let $\circ$ be a belief-change-operator that satisfies (R1) to (R5), (R7), (C1) and $\psi, \mu_1, \mu_2 \in \mathcal{L}$ such that $(\psi \circ \mu_1) \wedge (\psi \circ \mu_2)$ does not imply $\psi \circ (\mu_1 \vee \mu_2)$, then for every $\omega \in \mathrm{Mod}((\psi \circ \mu_1) \wedge (\psi \circ \mu_2))$ that is not in $\mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2))$ there exists a blocking set $\mathcal{N} \subseteq \Omega$.*

Proof. We define $\mathcal{N}_1 = \mathrm{Mod}(\psi \circ \mu_1)$ and $\mathcal{N}_2 = \mathrm{Mod}(\psi \circ \mu_2)$. With (R5) we can infer that $(\psi \circ (\mu_1 \vee \mu_2)) \wedge \mu_1$ implies $\psi \circ ((\mu_1 \vee \mu_2) \wedge \mu 1)$ which is equivalent to $\psi \circ \mu_1$ and therefore $\mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2)) \cap \mathrm{Mod}(\mu_1) \subseteq \mathrm{Mod}(\psi \circ \mu_1) = \mathcal{N}_1$ and with the same argument $\mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2)) \cap \mathrm{Mod}(\mu_2) \subseteq \mathrm{Mod}(\psi \circ \mu_2) = \mathcal{N}_2$. The intersection $\mathcal{N}_1 \cap \mathcal{N}_2$ is not empty because otherwise $\mathrm{Mod}(\psi \circ \mu_1) \cap \mathrm{Mod}(\psi \circ \mu_2) = \emptyset \subseteq \mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2))$. Let $\omega \in \mathcal{N}_1 \cap \mathcal{N}_2$ such that $\omega \notin \mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2))$, set $\mathcal{N}_1' = \mathcal{N}_1 \cap \mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2))$ and which is nonempty because otherwise $\mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2)) \subseteq \mathcal{N}_2$ and there would exist $\mathcal{N}' \subseteq \mathcal{N}_2$ such that $\psi \circ \mathrm{form}(\mathcal{N}', \omega) = \mathrm{form}(\mathcal{N}')$ which violates (C1). With the same argument $\mathcal{N}_2' = \mathcal{N}_2 \cap \mathrm{Mod}(\psi \circ (\mu_1 \vee \mu_2))$ is nonempty.

Starting with $\mathcal{N}_3 = \mathcal{N}_1 \cup \mathcal{N}_2$ we 'trim' until we have an iteration $\mathcal{N}_\mathrm{i} \subseteq \mathcal{N}_3$ for $\mathrm{i} \geqslant 3$ such that $\psi \circ \mathrm{form}(\mathcal{N}_\mathrm{i}, \omega) = \mathrm{form}(\mathcal{N}_\mathrm{i}, \omega)$. We do this in the following order:

For $\mathcal{N}_\mathrm{i} \subseteq \mathcal{N}_3$ we test for $\omega_\mathrm{i} \in \mathcal{N}_\mathrm{i}$ whether $\psi \circ \mathrm{form}(\mathcal{N}_\mathrm{i} \backslash \{\omega_\mathrm{i}\}, \omega) = \mathrm{form}(\mathcal{N}_\mathrm{i} \backslash \{\omega_\mathrm{i}\})$ and if that is the case we set $\mathcal{N}_{\mathrm{i}+1} = \mathcal{N}_\mathrm{i} \backslash \{\omega_\mathrm{i}\}$. We repeat this until such ω_i can no longer be found.

This terminates because $\mathcal{N}_3$ is finite and in the case that $|\mathcal{N}_\mathrm{i}| = 2$ it follows that any potential $\mathcal{N}_{\mathrm{i}+1}$ is subset of either $\mathcal{N}_1$ or $\mathcal{N}_2$ and therefore $\psi \circ \mathrm{form}(\mathcal{N}_{1+1}, \omega) = \mathrm{form}(\mathcal{N}_{1+1}, \omega)$. The last iteration is a blocking set because for all $\omega_\mathrm{i} \in \mathcal{N}_\mathrm{i}$ it follows that $\psi \circ \mathrm{form}(\mathcal{N}_\mathrm{i} \backslash \{\omega_\mathrm{i}\}, \omega) = \mathrm{form}(\mathcal{N}_\mathrm{i} \backslash \{\omega_\mathrm{i}\}, \omega)$ and with (C1) it follows that $\psi \circ \mathrm{form}(\mathcal{N}', \omega) = \mathrm{form}(\mathcal{N}', \omega)$ for all $\mathcal{N}' \subsetneq \mathcal{N}_\mathrm{i}$. □

A postulate that expresses the knowledge, that the size of blocking sets is limited by the size of $\mathrm{Mod}(\psi)$, has to be very specific. The following proposal shows this problem:

(C3) Let $\mathrm{n} \in \mathbb{N}$ such that $\psi \equiv (\psi_1 \vee ... \vee \psi_\mathrm{n})$, for $\mathrm{i} \neq \mathrm{j}$, $\mathrm{i}, \mathrm{j} \in \{1, ..., \mathrm{n}\}$ then $\psi_\mathrm{i} \wedge \psi_\mathrm{j}$ is not satisfiable, $\psi \wedge \psi_\mathrm{i}$ is satisfiable, and n is maximum. Let $\phi \equiv (\phi_1 \vee ... \vee \phi_\mathrm{m})$ such that $\phi_\mathrm{i} \wedge \phi_\mathrm{j}$ is not satisfiable and $\phi \wedge \phi_\mathrm{i}$ is satisfiable (for $\mathrm{i}, \mathrm{j} \in \{1, ..., \mathrm{m}\}$ and $\mathrm{i} \neq \mathrm{j}$), m is maximum, $\psi \circ (\mu \vee \phi) \equiv \phi$ and $\psi \circ (\mu \vee \phi_1 \vee ... \phi_{i-1} \vee \phi_{i+1} \vee ... \vee \phi_m) \equiv (\mu \vee \phi_1 \vee ... \phi_{i-1} \vee \phi_{i+1} \vee ... \vee \phi_m)$, then $\mathrm{m} \leq \mathrm{n}$.

It stands to reason that we could formulate a weak faithful cyclic assignment for a belief change operator satisfying (R1) to (R5), (R7), (C1), (C2) and (C3) the following way:

If $\mathrm{Mod}(\psi) \neq \emptyset$ then $\mathcal{M}_\psi = \mathrm{Mod}(\psi)$, else $\mathcal{M}_\psi = \mathrm{Mod}(\psi \circ \top)$. For all $\omega_\mathrm{i}, \omega_\mathrm{j} \in \Omega \backslash \mathcal{M}_\psi$ if $(\psi \circ \mathrm{form}(\omega_\mathrm{i}, \omega_\mathrm{j})) \equiv \mathrm{form}(\omega_\mathrm{i})$ then for all $\omega_\psi \in \mathcal{M}_\psi : (\omega_\psi, \omega_\mathrm{i}, \omega_\mathrm{j}) \in \mathcal{C}_\psi$. For the blocking sets we consider $\mathcal{P}(\Omega)$ and start with $\mathcal{N} \in \mathcal{P}(\Omega)$ such that $|\mathcal{N}| = 2$ and iterate until (and including) $|\mathcal{N}| = |\mathcal{M}_\psi|$. We simply check for all $\omega \notin \mathcal{N}$ whether $\mathcal{N}$ is a blocking set for it. If that is the case then for $\mathrm{k} = |\mathcal{N}|$ and $\mathrm{n} = |\mathcal{M}_\psi|$:

$$(\omega_{\psi_1}, \omega_1, \omega) \in \mathcal{C}_\psi,$$
$$\dots$$
$$(\omega_{\psi_\mathrm{k}}, \omega_\mathrm{k}, \omega) \in \mathcal{C}_\psi,$$
$$\dots$$
$$(\omega_{\psi_\mathrm{n}}, \omega_\mathrm{k}, \omega) \in \mathcal{C}_\psi.$$

Because of (R7) and (C2) we could simply add $(\omega_\psi, \omega_\mathrm{i}, \omega_\mathrm{k}) \in \mathcal{C}_\psi$ and $(\omega_\mathrm{i}, \omega_\mathrm{j}, \omega_\mathrm{k})$, if $(\omega_\psi, \omega_\mathrm{i}, \omega_\mathrm{j}) \in \mathcal{C}_\psi$ and $(\omega_\psi, \omega_\mathrm{j}, \omega_\mathrm{k}) \in \mathcal{C}_\psi$. After that we could simply add the cyclic permutations.

If then some $\omega \in \mathrm{Mod}(\mu)$ is close to $\mathcal{M}_\psi$ in $\mathcal{C}_\psi$ with respect to $\mathrm{Mod}(\mu)$ it would easily follow that $\omega \in \mathrm{Mod}(\psi \circ \mu)$.

However with multiple blocking sets, this method would surely create additional blocking sets that are not part of the initial revision operator. Therefore we cannot use this method. Instead we have to further investigate the interdependence of blocking sets and the limits of weak faithful cyclic orders to represent them.

While we know that the size of a blocking set is restricted by the size of $\mathrm{Mod}(\psi)$, or if $\mathrm{Mod}(\psi) = \emptyset$ then by the size of $\mathcal{M}_\psi$, we have to consider how multiple blocking sets for some ω, that do not share any elements, interfere with each other. From the side of the belief-change-operator, it could be feasible to have two blocking sets who cannot be combined into a third blocking set. To illustrate this, we look at the following example:

Example 11. *We define the problem formally: Let $\mathcal{N}_1$ and $\mathcal{N}_2$ two blocking sets for ω such that $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$ and there exists no $\mathcal{N}_3 \subseteq \mathcal{N}_1 \cup \mathcal{N}_2$ such that $\mathcal{N}_3 \neq \mathcal{N}_1 \neq \mathcal{N}_2 \neq \mathcal{N}_3$ is a blocking set for ω. Then the problem lies in the question whether or not we can map this change-operator to a weak faithful cyclic order.*

The following directed graph represents the part of interest of the cyclic order $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_1}, \omega_4, \omega), (\omega_{\psi_2}, \omega_2, \omega), (\omega_{\psi_2}, \omega_3, \omega), (\omega_{\psi_3}, \omega_3, \omega), (\omega_{\psi_3}, \omega_1, \omega), (\omega_{\psi_4}, \omega_4, \omega), (\omega_{\psi_4}, \omega_2, \omega)\}$: Here $\mathcal{N}_1 = \{\omega_1, \omega_2\}$ and $\mathcal{N}_2 = \{\omega_3, \omega_4\}$ are blocking sets for ω and:

1. *$\mathcal{N}_3 = \{\omega_1, \omega_3\}$ is not a blocking set because neither $(\omega_{\psi_4}, \omega_1, \omega) \in \mathcal{C}_\psi$ nor $(\omega_{\psi_4}, \omega_3, \omega) \in \mathcal{C}_\psi$.*
2. *$\mathcal{N}_4 = \{\omega_1, \omega_4\}$ is not a blocking set because neither $(\omega_{\psi_2}, \omega_1, \omega) \in \mathcal{C}_\psi$ nor $(\omega_{\psi_2}, \omega_4, \omega) \in \mathcal{C}_\psi$.*
3. *$\mathcal{N}_5 = \{\omega_2, \omega_3\}$ is not a blocking set because neither $(\omega_{\psi_1}, \omega_2, \omega) \in \mathcal{C}_\psi$ nor $(\omega_{\psi_1}, \omega_3, \omega) \in \mathcal{C}_\psi$.*
4. *$\mathcal{N}_6 = \{\omega_2, \omega_4\}$ is not a blocking set because neither $(\omega_{\psi_3}, \omega_2, \omega) \in \mathcal{C}_\psi$ nor $(\omega_{\psi_3}, \omega_4, \omega) \in \mathcal{C}_\psi$.*

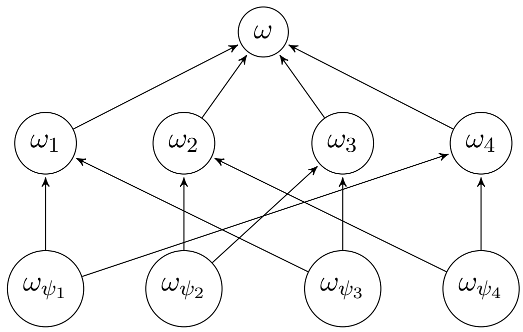

Figure 14: Part of the cyclic order in example 11

We see that a belief-change-operator with the blocking sets $\mathcal{N}_1$, $\mathcal{N}_2$ can be mapped to a weak faithful cyclic order (if $|\mathrm{Mod}(\psi)| = 4$) without the creation of an additional blocking set in the cyclic order.

However there is a problem in the case that a belief-change-operator $\circ$ has two distinct blocking sets $\mathcal{N}_1$ and $\mathcal{N}_2$ for $\omega \in \Omega$ such that $|\mathcal{N}_1| + |\mathcal{N}_2| > |\mathrm{Mod}(\psi)| > \max_{\mathrm{i} \in \{1,2\}} |\mathcal{N}_\mathrm{i}|$ and $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$. Intuitively a weak faithful cyclic order cannot represent this case, because a $\mathcal{N}_3 \subseteq \mathcal{N}_1 \cup \mathcal{N}_2$, such that $\mathcal{N}_1 \neq \mathcal{N}_3 \neq \mathcal{N}_2$ and $\mathcal{N}_3$ is a blocking set for ω, should be possible. Therefore we formulate the following lemma:

Lemma 20. *Let $|\mathcal{M}_\psi| = \mathrm{n}$, $\mathcal{C}_\psi$ a weak faithful cyclic order, $\circ$ the belief-change-operator received from $\mathcal{C}_\psi$, $\mathcal{N}_1 \text{ and } \mathcal{N}_2$ blocking sets for $\omega \in \Omega$ such that $|\mathcal{N}_1| + |\mathcal{N}_2| > |\mathcal{M}_\psi| \geqslant \max_{\mathrm{i} \in \{1,2\}} |\mathcal{N}_\mathrm{i}| \geqslant \min_{\mathrm{i} \in \{1,2\}} |\mathcal{N}_\mathrm{i}| > 1$ and $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$. Then there exists $\mathcal{N}_3 \subseteq \mathcal{N}_1 \cup \mathcal{N}_2$, such that $\mathcal{N}_1 \neq \mathcal{N}_3 \neq \mathcal{N}_2$ and $\mathcal{N}_3$ is a blocking set for ω.*

Proof. We know that every ω_i of a blocking set has a world $\omega_{\psi_\mathrm{i}} \in \mathcal{M}_\psi$ that is exclusive to ω_i with respect to the blocking set, in the sense that no $\omega_j \neq \omega_i$ of the blocking set satisfies $(\omega_{\psi_\mathrm{i}}, \omega_\mathrm{j}, \omega) \in \mathcal{C}_\psi$. Because $|\mathcal{N}_1| + |\mathcal{N}_2| > |\mathcal{M}_\psi| \geqslant \max_{\mathrm{i} \in \{1,2\}} |\mathcal{N}_\mathrm{i}|$ it follows that some $\omega_1 \in \mathcal{N}_1$ and $\omega_2 \in \mathcal{N}_2$ share their exclusive ω_{ψ_i}. We can build a set $\mathcal{N}' = (\mathcal{N}_1 \backslash \{\omega_1\}) \cup \{\omega_2\}$ which is not necessarily a blocking set, because ω_1 can 'block' $\omega_{\psi_\mathrm{j}} \in \mathcal{M}_\psi$ that are exclusive to $\omega_\mathrm{j} \in \mathcal{N}_2$ with respect to $\mathcal{N}_2$, in the sense that $(\omega_{\psi_\mathrm{j}}, \omega_1, \omega))$. We can define $\mathcal{M}_{\omega_1} = \{\omega_{\psi_\mathrm{j}} | \omega_{\psi_\mathrm{j}} \in \mathcal{M}_\psi \text{ and } (\omega_{\psi_\mathrm{j}}, \omega_1, \omega) \in \mathcal{C}_\psi\}$ and for all elements in $\mathcal{M}^\emptyset_{\omega_1} = \{\omega_{\psi_\mathrm{j}} \in \mathcal{M}_{\omega_1} | \forall \omega_\mathrm{i} \in \mathcal{N}' : (\omega_{\psi_\mathrm{j}}, \omega_\mathrm{i}, \omega) \notin \mathcal{C}_\psi\}$, we could simply add the corresponding $\tilde{\omega}_2 \in \mathcal{N}_2$. However we have to observe the extreme case $\mathcal{N}_2 \subseteq \mathcal{M}^\emptyset_{\omega_1}$ which can be resolved by changing the roles of ω_1 and ω_2. If $\mathcal{N}_1 \subseteq \mathcal{M}^\emptyset_{\omega_2}$ then the set $\{\omega_1, \omega_2\}$ is a blocking set. Otherwise the set $\mathcal{N}_3 = \mathcal{N}' \cup \{\omega_\mathrm{j} \in \mathcal{N}_2 | \, \omega_{\psi_\mathrm{j}} \in \mathcal{M}^\emptyset_{\omega_1} \text{ is exclusive to } \omega_\mathrm{j}\}$ can be 'trimmed' down to a blocking set. ☐

While we could try to generalize this to multiple blocking sets $\mathcal{N}_1, ..., \mathcal{N}_\mathrm{n}$ with $\sum_{\mathrm{i} \in \{1,...,\mathrm{n}\}} |\mathcal{N}_\mathrm{i}| > |\mathcal{M}_\psi|$, we have not answered the question whether two individual

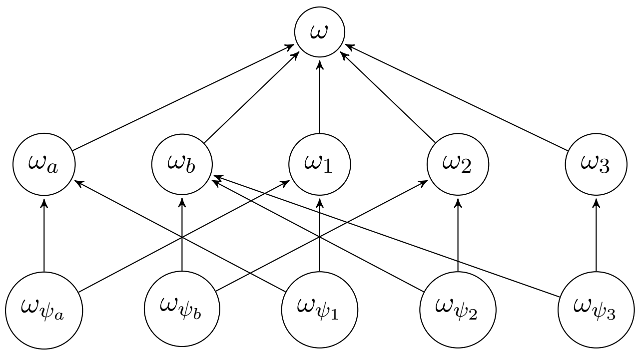

Figure 15: Case (a | 1), (b | 2,3), (1 | a), (2 | b), (3 | $\emptyset$) in proof for lemma 21

blocking sets $|\mathcal{N}_1| + |\mathcal{N}_2| \leqslant |\mathcal{M}_\psi|$ can always be mapped to a weak faithful cyclic order (without creating an additional blocking set). In example 11 only one case is shown. The obvious next case consists of two blocking sets $|\mathcal{N}_1| = 2$ and $|\mathcal{N}_2| = 3$ for $|\mathcal{M}_\psi| = 5$, however the following lemma shows that a mapping to a weak faithful cyclic order (in this case) always leads to an additional blocking set:

Lemma 21. *Let $\circ$ be a belief-change-operator that satisfies (R1) to (R5), (R7), (C1), (C2), $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$ be blocking sets for $\omega \in \Omega$ with $|\mathcal{N}_1| = 2$, $|\mathcal{N}_2| = 3$ and $|\mathcal{M}_\psi| = 5$. If $\mathcal{N}_1$ and $\mathcal{N}_2$ are the only blocking sets for ω then $\circ$ cannot be mapped to a weak faithful cyclic order.*

Proof. We could combine all possible combinations of $\mathcal{N}_1$ and $\mathcal{N}_2$ blocking ω in a cyclic order. Here our only constraint is the fact that every ω_i in a blocking set needs $\omega_{\psi_j} \in \mathcal{M}_\psi$ that is exclusive to ω_i with respect to the blocking set. However the number of possible cases makes it unreasonable to pursue this method. Instead we want to only consider the cases where we identify each of the $\omega_{\psi_j} \in \mathcal{M}_\psi$ by one $\omega_j \in \mathcal{N}_1 \cup \mathcal{N}_2$. The reason for that lies in the arguments presented in the proof of lemma 20.

We therefore define $\mathcal{N}_1 = \{\omega_\mathrm{a}, \omega_\mathrm{b}\}$, $\mathcal{N}_2 = \{\omega_1, \omega_2, \omega_3\}$ and identify $\mathcal{M}_\psi = \{\omega_{\psi_\mathrm{a}}, \omega_{\psi_\mathrm{b}}, \omega_{\psi_1}, \omega_{\psi_2}, \omega_{\psi_3}\}$. For ω_a we note (a | 1,2) if $\{(\omega_{\psi_\mathrm{a}}, \omega_\mathrm{a}, \omega), (\omega_{\psi_1}, \omega_\mathrm{a}, \omega), (\omega_{\psi_2}, \omega_\mathrm{a}, \omega)\} \subseteq \mathcal{C}_\psi$. The benefit of this codification lies in the easy decoding as the following example demonstrates: We see that (b | 2,3) and (1 | a) cover all $\omega_{\psi_\mathrm{j}} \in \mathcal{M}_\psi$ and therefore $\mathcal{N}_3 = \{\omega_\mathrm{b}, \omega_1\}$ is a blocking set for ω. When we write (x | ...) the x encodes both ω_x and the exclusive ω_{ψ_x}.

In the tables 1 and 2 we list the basic cases. It is important to note that these are indeed basic and sufficient because for (a | 1), (b | 2,3) we could also consider (a | 1,2), (b | 2,3) which would only simplify the search for an additional blocking set $\mathcal{N}_3$. We need to only consider the minimal cases where $\mathcal{N}_1$ and $\mathcal{N}_2$ are blocking sets. As a result we see that for every possible basic mapping an additional blocking set $\mathcal{N}_3$ can be found. Therefore we cannot map $\circ$ to a weak faithful cyclic order if $\mathcal{N}_1$ and $\mathcal{N}_2$ are the only blocking sets for ω. $\square$

ω_a	ω_b	ω_1	ω_2	ω_3	$\mathcal{N}_3$
(a | $\emptyset$)	(b | 1,2,3)	(1 | $\emptyset$)	(2 | $\emptyset$)	(3 | a,b)	(b | 1,2,3), (3 | a,b)
(a | $\emptyset$)	(b | 1,2,3)	(1 | $\emptyset$)	(2 | a)	(3 | b)	(b | 1,2,3), (2 | a)
(a | $\emptyset$)	(b | 1,2,3)	(1 | $\emptyset$)	(2 | b)	(3 | a)	(b | 1,2,3), (3 | a)
(a | $\emptyset$)	(b | 1,2,3)	(1 | $\emptyset$)	(2 | a,b)	(3 | $\emptyset$)	(b | 1,2,3), (2 | a,b)
(a | $\emptyset$)	(b | 1,2,3)	(1 | a)	(2 | $\emptyset$)	(3 | b)	(b | 1,2,3), (1 | a)
(a | $\emptyset$)	(b | 1,2,3)	(1 | a)	(2 | b)	(3 | $\emptyset$)	(b | 1,2,3), (1 | a)
(a | $\emptyset$)	(b | 1,2,3)	(1 | b)	(2 | $\emptyset$)	(3 | a)	(b | 1,2,3), (3 | a)
(a | $\emptyset$)	(b | 1,2,3)	(1 | b)	(2 | a)	(3 | $\emptyset$)	(b | 1,2,3), (2 | a)
(a | $\emptyset$)	(b | 1,2,3)	(1 | a,b)	(2 | $\emptyset$)	(3 | $\emptyset$)	(b | 1,2,3), (1 | a,b)
(a | 1)	(b | 2,3)	(1 | $\emptyset$)	(2 | $\emptyset$)	(3 | a,b)	(b | 2,3), (1 | $\emptyset$), (3 | a,b)
(a | 1)	(b | 2,3)	(1 | $\emptyset$)	(2 | a)	(3 | b)	(b | 2,3), (1 | $\emptyset$), (2 | a)
(a | 1)	(b | 2,3)	(1 | $\emptyset$)	(2 | b)	(3 | a)	(b | 2,3), (1 | $\emptyset$), (3 | a)
(a | 1)	(b | 2,3)	(1 | $\emptyset$)	(2 | a,b)	(3 | $\emptyset$)	(a | 1), (2 | $\emptyset$), (3 | b)
(a | 1)	(b | 2,3)	(1 | a)	(2 | $\emptyset$)	(3 | b)	(b | 2,3), (1 | a)
(a | 1)	(b | 2,3)	(1 | a)	(2 | b)	(3 | $\emptyset$)	(b | 2,3), (1 | a)
(a | 1)	(b | 2,3)	(1 | b)	(2 | $\emptyset$)	(3 | a)	(b | 2,3), (1 | b), (3 | a)
(a | 1)	(b | 2,3)	(1 | b)	(2 | a)	(3 | $\emptyset$)	(b | 2,3), (1 | b), (2 | a)
(a | 1)	(b | 2,3)	(1 | a,b)	(2 | $\emptyset$)	(3 | $\emptyset$)	(b | 2,3), (1 | a,b)
(a | 2)	(b | 1,3)	(1 | $\emptyset$)	(2 | $\emptyset$)	(3 | a,b)	(b | 1,3), (2 | $\emptyset$), (3 | a,b)
(a | 2)	(b | 1,3)	(1 | $\emptyset$)	(2 | a)	(3 | b)	(b | 1,3), (2 | a)
(a | 2)	(b | 1,3)	(1 | $\emptyset$)	(2 | b)	(3 | a)	(b | 1,3), (2 | b), (3 | a)
(a | 2)	(b | 1,3)	(1 | $\emptyset$)	(2 | a,b)	(3 | $\emptyset$)	(b | 1,3), (2 | a,b)
(a | 2)	(b | 1,3)	(1 | a)	(2 | $\emptyset$)	(3 | b)	(b | 1,3), (1 | a), (2 | $\emptyset$)
(a | 2)	(b | 1,3)	(1 | a)	(2 | b)	(3 | $\emptyset$)	(b | 1,3), (1 | a), (2 | b)
(a | 2)	(b | 1,3)	(1 | b)	(2 | $\emptyset$)	(3 | a)	(b | 1,3), (2 | $\emptyset$), (3 | a)
(a | 2)	(b | 1,3)	(1 | b)	(2 | a)	(3 | $\emptyset$)	(b | 1,3), (2 | a)
(a | 2)	(b | 1,3)	(1 | a,b)	(2 | $\emptyset$)	(3 | $\emptyset$)	(a | 2), (1 | a,b), (3 | $\emptyset$)
(a | 3)	(b | 1,2)	(1 | $\emptyset$)	(2 | $\emptyset$)	(3 | a,b)	(b | 1,2), (3 | a,b)
(a | 3)	(b | 1,2)	(1 | $\emptyset$)	(2 | a)	(3 | b)	(b | 1,2), (2 | a), (3 | b)
(a | 3)	(b | 1,2)	(1 | $\emptyset$)	(2 | b)	(3 | a)	(b | 1,2), (3 | a)
(a | 3)	(b | 1,2)	(1 | $\emptyset$)	(2 | a,b)	(3 | $\emptyset$)	(b | 1,2), (2 | a,b), (3 | $\emptyset$)
(a | 3)	(b | 1,2)	(1 | a)	(2 | $\emptyset$)	(3 | b)	(b | 1,2), (1 | a), (3 | b)
(a | 3)	(b | 1,2)	(1 | a)	(2 | b)	(3 | $\emptyset$)	(b | 1,2), (1 | a), (3 | $\emptyset$)
(a | 3)	(b | 1,2)	(1 | b)	(2 | $\emptyset$)	(3 | a)	(b | 1,2), (3 | a)
(a | 3)	(b | 1,2)	(1 | b)	(2 | a)	(3 | $\emptyset$)	(b | 1,2), (2 | a), (3 | $\emptyset$)
(a | 3)	(b | 1,2)	(1 | a,b)	(2 | $\emptyset$)	(3 | $\emptyset$)	(a | 3), (1 | a,b), (2 | $\emptyset$)

Table 1: Cases proof lemma 21 page 1

ω_a	ω_b	ω_1	ω_2	ω_3	$\mathcal{N}_3$
(a | 1,2)	(b | 3)	(1 | ∅)	(2 | ∅)	(3 | a,b)	(a | 1,2), (3 | a,b)
(a | 1,2)	(b | 3)	(1 | ∅)	(2 | a)	(3 | b)	(a | 1,2), (3 | b)
(a | 1,2)	(b | 3)	(1 | ∅)	(2 | b)	(3 | a)	(a | 1,2), (2 | b), (3 | a)
(a | 1,2)	(b | 3)	(1 | ∅)	(2 | a,b)	(3 | ∅)	(b | 3), (1 | ∅), (2 | a,b)
(a | 1,2)	(b | 3)	(1 | a)	(2 | ∅)	(3 | b)	(a | 1,2), (3 | b)
(a | 1,2)	(b | 3)	(1 | a)	(2 | b)	(3 | ∅)	(a | 1,2), (2 | b), (3 | ∅)
(a | 1,2)	(b | 3)	(1 | b)	(2 | ∅)	(3 | a)	(a | 1,2), (1,b), (3 | a)
(a | 1,2)	(b | 3)	(1 | b)	(2 | a)	(3 | ∅)	(b | 3), (1 | b), (2 | a)
(a | 1,2)	(b | 3)	(1 | a,b)	(2 | ∅)	(3 | ∅)	(b | 3), (1 | a,b), (2 | ∅)
(a | 1,3)	(b | 2)	(1 | ∅)	(2 | ∅)	(3 | a,b)	(b | 2), (1 | ∅), (3 | a,b)
(a | 1,3)	(b | 2)	(1 | ∅)	(2 | a)	(3 | b)	(a | 1,3), (2 | a), (3 | b)
(a | 1,3)	(b | 2)	(1 | ∅)	(2 | b)	(3 | a)	(a | 1,3), (2 | b)
(a | 1,3)	(b | 2)	(1 | ∅)	(2 | a,b)	(3 | ∅)	(a | 1,3), (2 | a,b)
(a | 1,3)	(b | 2)	(1 | a)	(2 | ∅)	(3 | b)	(a | 1,3), (2 | ∅), (3 | b)
(a | 1,3)	(b | 2)	(1 | a)	(2 | b)	(3 | ∅)	(a | 1,3), (2 | b)
(a | 1,3)	(b | 2)	(1 | b)	(2 | ∅)	(3 | a)	(a | 1,3), (1 | b), (2 | ∅)
(a | 1,3)	(b | 2)	(1 | b)	(2 | a)	(3 | ∅)	(a | 1,3), (1 | b), (2 | a)
(a | 1,3)	(b | 2)	(1 | a,b)	(2 | ∅)	(3 | ∅)	(a | 1,3), (1 | a,b), (2 | ∅)
(a | 2,3)	(b | 1)	(1 | ∅)	(2 | ∅)	(3 | a,b)	(b | 1), (2 | ∅), (3 | a,b)
(a | 2,3)	(b | 1)	(1 | ∅)	(2 | a)	(3 | b)	(a | 2,3), (1 | ∅), (3 | b)
(a | 2,3)	(b | 1)	(1 | ∅)	(2 | b)	(3 | a)	(a | 2,3), (1 | ∅), (2 | b)
(a | 2,3)	(b | 1)	(1 | ∅)	(2 | a,b)	(3 | ∅)	(b | 1), (2 | a,b), (3 | ∅)
(a | 2,3)	(b | 1)	(1 | a)	(2 | ∅)	(3 | b)	(a | 2,3), (1 | a), (3 | b)
(a | 2,3)	(b | 1)	(1 | a)	(2 | b)	(3 | ∅)	(a | 2,3), (1 | a), (2 | b)
(a | 2,3)	(b | 1)	(1 | b)	(2 | ∅)	(3 | a)	(a | 2,3), (1 | b)
(a | 2,3)	(b | 1)	(1 | b)	(2 | a)	(3 | ∅)	(a | 2,3), (1 | b)
(a | 2,3)	(b | 1)	(1 | a,b)	(2 | ∅)	(3 | ∅)	(a | 2,3), (1 | a,b)
(a | 1,2,3)	(b | ∅)	(1 | ∅)	(2 | ∅)	(3 | a,b)	(a | 1,2,3), (3 | a,b)
(a | 1,2,3)	(b | ∅)	(1 | ∅)	(2 | a)	(3 | b)	(a | 1,2,3), (3 | b)
(a | 1,2,3)	(b | ∅)	(1 | ∅)	(2 | b)	(3 | a)	(a | 1,2,3), (2 | b)
(a | 1,2,3)	(b | ∅)	(1 | ∅)	(2 | a,b)	(3 | ∅)	(a | 1,2,3), (2 | a,b)
(a | 1,2,3)	(b | ∅)	(1 | a)	(2 | ∅)	(3 | b)	(a | 1,2,3), (3 | b)
(a | 1,2,3)	(b | ∅)	(1 | a)	(2 | b)	(3 | ∅)	(a | 1,2,3), (2 | b)
(a | 1,2,3)	(b | ∅)	(1 | b)	(2 | ∅)	(3 | a)	(a | 1,2,3), (1 | b)
(a | 1,2,3)	(b | ∅)	(1 | b)	(2 | a)	(3 | ∅)	(a | 1,2,3), (1 | b)
(a | 1,2,3)	(b | ∅)	(1 | a,b)	(2 | ∅)	(3 | ∅)	(a | 1,2,3), (1 | a,b)

Table 2: Cases proof lemma 21 page 2

$\mathcal{N}_3$	not 'blocked'
$\{\omega_{\mathrm{a}}, \omega_1, \omega_2\}$	ω_{ψ_6}
$\{\omega_{\mathrm{a}}, \omega_1, \omega_3\}$	ω_{ψ_5}
$\{\omega_{\mathrm{a}}, \omega_2, \omega_3\}$	ω_{ψ_4}
$\{\omega_{\mathrm{b}}, \omega_1, \omega_2\}$	ω_{ψ_3}
$\{\omega_{\mathrm{b}}, \omega_1, \omega_3\}$	ω_{ψ_2}
$\{\omega_{\mathrm{b}}, \omega_2, \omega_3\}$	ω_{ψ_1}

Table 3: Candidates for an additional blocking set in example 12

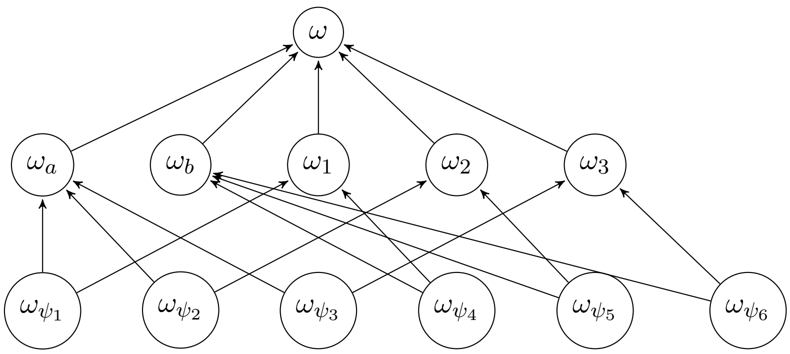

Figure 16: Part of the cyclic order in example 12

This however does not show that a mapping of the blocking sets $\mathcal{N}_1$ and $\mathcal{N}_2$ for ω to a cyclic order, without the creation of an additional blocking set in the cyclic order, is impossible:

Example 12. *If $|\mathcal{M}_\psi| = 6$ we can construct the cyclic order $\mathcal{C}_\psi = \{(\omega_{\psi_1}, \omega_{\mathrm{a}}, \omega), (\omega_{\psi_2}, \omega_{\mathrm{a}}, \omega), (\omega_{\psi_3}, \omega_{\mathrm{a}}, \omega), (\omega_{\psi_4}, \omega_{\mathrm{b}}, \omega), (\omega_{\psi_5}, \omega_{\mathrm{b}}, \omega), (\omega_{\psi_6}, \omega_{\mathrm{b}}, \omega),$ $(\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_4}, \omega_1, \omega), (\omega_{\psi_2}, \omega_2, \omega), (\omega_{\psi_5}, \omega_2, \omega), (\omega_{\psi_3}, \omega_3, \omega), (\omega_{\psi_6}, \omega_3, \omega)\}$ where $\mathcal{N}_1 = \{\omega_{\mathrm{a}}, \omega_{\mathrm{b}}\}$ and $\mathcal{N}_2 = \{\omega_1, \omega_2, \omega_3\}$ are blocking sets for ω and there is no other blocking set for ω. To show this we need to only look at the candidates in table 3.*

Because we have found mappings in the two cases $|\mathcal{N}_1| = 2,\ |\mathcal{N}_2| = 2,\ |\mathcal{M}_\psi| = 4$ and $|\mathcal{N}_1| = 2,\ |\mathcal{N}_2| = 3,\ |\mathcal{M}_\psi| = 6$ it is reasonable to assume that this is always the case if $|\mathcal{M}_\psi| = |\mathcal{N}_1| * |\mathcal{N}_2|$. We formulate the following lemma accordingly:

Lemma 22. *Let $\mathcal{N}_1 \subseteq \Omega$ and $\mathcal{N}_2 \subseteq \Omega$ such that $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$, $|\mathcal{N}_1| = \mathrm{n}$ and $|\mathcal{N}_2| = \mathrm{m}$. If $|\mathcal{M}_\psi| = \mathrm{n} * \mathrm{m}$ and there exists some $\omega \in \Omega \backslash (\mathcal{M}_\psi \cup \mathcal{N}_1 \cup \mathcal{N}_2)$, then there exists a weak faithful cyclic order C_ψ such that $\mathcal{N}_1$ and $\mathcal{N}_2$ are the only blocking sets for ω.*

Proof. Because all three sets are finite we can list their elements and demand the fix index sets $\mathrm{I} = \{1, ..., \mathrm{n}\}$ for $\mathcal{N}_1 = \{\omega_{1_1}, ..., \omega_{1_\mathrm{n}}\}$, $\mathrm{J} = \{1, ..., \mathrm{m}\}$ for $\mathcal{N}_2 = \{\omega_{2_1}, ..., \omega_{2_\mathrm{m}}\}$ and $\mathrm{H} = \{1, ..., \mathrm{n} * \mathrm{m}\}$ for $\mathcal{M}_\psi = \{\omega_{\psi_1}, ..., \omega_{\psi_{(\mathrm{n}*\mathrm{m})}}\}$.

For some $\omega \notin \mathcal{M}_\psi \cup \mathcal{N}_1 \cup \mathcal{N}_2$ we create the cyclic order $\mathcal{C}_\psi = \bigcup_{1 \leqslant i \leqslant n} \{(\omega_{\psi_{((i-1)*m+j)}}, \omega_{1_i}, \omega) | j \in J\} \cup \bigcup_{1 \leqslant j \leqslant m} \{(\omega_{\psi_{((i-1)*m+j)}}, \omega_{2_j}, \omega) | i \in I\}$.

We see that $\mathrm{closest}(\mathcal{N}_1 \cup \{\omega\}, \mathcal{M}_\psi, \mathcal{C}_\psi) = \mathcal{N}_1$ because for all $\omega_{\psi_h} \in \mathcal{M}_\psi$ we can define $i = \lceil h/m \rceil$, $j = h \bmod m$ and per definition $(\omega_{\psi_{((i-1)*m+j)}}, \omega_{1_i}, \omega) \in \mathcal{C}_\psi$. Moreover for all $\omega_{1_i} \in \mathcal{N}_1, i \in I$ it follows that $\mathrm{closest}(\mathcal{N}_1 \backslash \{\omega_{1_i}\} \cup \{\omega\}, \mathcal{M}_\psi, \mathcal{C}_\psi) = \mathcal{N}_1 \backslash \{\omega_{1_i}\} \cup \{\omega\}$ because for $\omega_{\psi_{(i*m)}}$ there is no $\omega_{1_i} \neq \omega_{1_h} \in \mathcal{N}_1$ ($h \in I$) such that $(\omega_{\psi_{i*m}}, \omega_{1_h}, \omega) \in \mathcal{C}_\psi$. Therefore $\mathcal{N}_1$ is a blocking set for ω.

The same argument can be used to show that $\mathcal{N}_2$ is a blocking set for ω too. Then only difference lies in the used indices.

We need to show that we cannot find an additional set $\mathcal{N}_3 \subseteq \mathcal{N}_1 \cup \mathcal{N}_2$ such that $\mathcal{N}_1 \nsubseteq \mathcal{N}_3$, $\mathcal{N}_2 \nsubseteq \mathcal{N}_3$ and $\mathcal{N}_3$ is a blocking set for ω. Towards contradiction we assume that such a $\mathcal{N}_3$ exists. Let $\omega_{1_i} \in \mathcal{N}_1$, $i \in I$ such that $\omega_{1_i} \notin \mathcal{N}_3$ then for $\omega_{\psi_{((i-1)*m+1)}}, ..., \omega_{\psi_{((i-1)*m+m)}}$ there need to exist $\tilde{\omega}_1, ..., \tilde{\omega}_m \in \mathcal{N}_3$ such that $(\omega_{\psi_{((i-1)*m+h)}}, \tilde{\omega}_h, \omega) \in \mathcal{C}_\psi$, for $h \in \{1, ..., m\}$. Because of our construction of $\mathcal{C}_\psi$ it follows that these $\tilde{\omega}_h \notin \mathcal{N}_1$ and therefore $\tilde{\omega}_h \in \mathcal{N}_2$. That implies $\mathcal{N}_2 \subseteq \mathcal{N}_3$ which is a contradiction. Therefore such $\mathcal{N}_3$ does not exist. □

Theorem 23. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ blocking sets for ω in a weak faithful cyclic order $\mathcal{C}_\psi$ such that $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$, $\max_{i \in \{1,2\}} |\mathcal{N}_i| > 2$, $\min_{i \in \{1,2\}} |\mathcal{N}_i| \geqslant 2$ and $|\mathcal{M}_\psi| < |\mathcal{N}_1| * |\mathcal{N}_2|$ then there exists $\mathcal{N}_3 \subseteq \mathcal{N}_1 \cup \mathcal{N}_2$, $\mathcal{N}_1 \neq \mathcal{N}_3 \neq \mathcal{N}_2$ and $\mathcal{N}_3$ is a blocking set for ω in $\mathcal{C}_\psi$.*

Proof. Towards contradiction we assume that $\mathcal{N}_1$ and $\mathcal{N}_2$ are blocking sets for $\omega \in \Omega$ such that $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$, $|\mathcal{N}_1| * |\mathcal{N}_2| > |\mathcal{M}_\psi|$ and there exists no $\mathcal{N}_3 \subsetneq \mathcal{N}_1 \cup \mathcal{N}_2$ such that $\mathcal{N}_1 \neq \mathcal{N}_3 \neq \mathcal{N}_2$ and $\mathcal{N}_3$ is a blocking set for ω. For all $\omega_i \in \mathcal{N}_1$ we define $\mathcal{M}_1^i = \{\omega_\psi \in \mathcal{M}_\psi | (\omega_\psi, \omega_i, \omega) \in \mathcal{C}_\psi \text{ and } \forall \omega_h \in \mathcal{N}_1 \backslash \{\omega_i\} : (\omega_\psi, \omega_h, \omega) \notin \mathcal{C}_\psi\}$. It is obvious that for all $\omega_i, \omega_j \in \mathcal{N}_1$ if $\omega_i \neq \omega_j$ then $\mathcal{M}_1^i \cap \mathcal{M}_1^j = \emptyset$ and therefore $(\sum_{\omega_i \in \mathcal{N}_1} |\mathcal{M}_1^i|) \leqslant |\mathcal{M}_\psi|$. The $\omega_\psi \in \mathcal{M}_1^i$ are the worlds of $\mathcal{M}_\psi$ that are 'exclusively blocked' by ω_i in the blocking set $\mathcal{N}_1$ for ω. As a next step we define for $\omega_i \in \mathcal{N}_1$ and $\omega_j \in \mathcal{N}_2$ a set $\mathcal{M}_\psi^{(i,j)} = \{\omega_\psi \in \mathcal{M}_1^i | (\omega_\psi, \omega_j, \omega) \in \mathcal{C}_\psi \text{ and } \forall \omega_h \in \mathcal{N}_2 \backslash \{\omega_j\} : (\omega_\psi, \omega_h, \omega) \notin \mathcal{C}_\psi\}$. Suppose one of these $\mathcal{M}_\psi^{(i,j)}$ is empty, then ω_j is not needed to block all of the $\omega_\psi \in \mathcal{M}_1^i$ i.e. for all $\omega_\psi \in \mathcal{M}_1^i$ there exists $\omega_h \in \mathcal{N}_2 \backslash \{\omega_j\}$ such that $(\omega_\psi, \omega_h, \omega) \in \mathcal{C}_\psi$. Therefore we can set $\mathcal{N}' = (\mathcal{N}_1 \cup \mathcal{N}_2) \backslash \{\omega_i, \omega_j\}$ and because $\psi \circ \mathrm{form}(\mathcal{N}', \omega) = \mathrm{form}(\mathcal{N}')$ there exists a blocking set $\mathcal{N}_3 \subsetneq \mathcal{N}_1 \cup \mathcal{N}_2$ and $\mathcal{N}_1 \neq \mathcal{N}_3 \neq \mathcal{N}_2$ (we can 'trim' $\mathcal{N}'$ until we receive such a set). This would be a contradiction and therefore none of the $\mathcal{M}_\psi^{(i,j)}$ is empty. These sets are disjoint for a fixed $\omega_i \in \mathcal{N}_1$ and therefore $|\mathcal{N}_2| \leqslant |\mathcal{M}_1^i|$ and as a consequence $|\mathcal{N}_1| * |\mathcal{N}_2| \leqslant (\sum_{\omega_i \in \mathcal{N}_1} |\mathcal{M}_1^i|) \leqslant |\mathcal{M}_\psi|$ which is a contradiction.

That means there exists $\mathcal{N}_3 \subseteq \mathcal{N}_1 \cup \mathcal{N}_2$, $\mathcal{N}_1 \neq \mathcal{N}_3 \neq \mathcal{N}_2$ and $\mathcal{N}_3$ is a blocking set for ω. □

A postulate that incorporates this theorem is the following postulate (C4):

$\mathcal{N}'$	does not block
$\{\omega_a, \omega_1, \omega_I\}$	ω_{ψ_8}
$\{\omega_a, \omega_1, \omega_{II}\}$	ω_{ψ_6}
$\{\omega_a, \omega_2, \omega_I\}$	ω_{ψ_7}
$\{\omega_a, \omega_2, \omega_{II}\}$	ω_{ψ_5}
$\{\omega_b, \omega_1, \omega_I\}$	ω_{ψ_4}
$\{\omega_b, \omega_1, \omega_{II}\}$	ω_{ψ_2}
$\{\omega_b, \omega_2, \omega_I\}$	ω_{ψ_3}
$\{\omega_b, \omega_2, \omega_{II}\}$	ω_{ψ_1}

Table 4: Candidates for an additional blocking set in example 13

(C4) Let $n \in \mathbb{N}$ such that $\psi \equiv (\psi_1 \vee ... \vee \psi_n)$, for $i \neq j$, $i, j \in \{1, ..., n\}$ then $\psi_i \wedge \psi_j$ is not satisfiable, $\psi \wedge \psi_i$ is satisfiable, and n is maximum. Let $\phi \equiv (\phi_1 \vee ... \vee \phi_m)$ such that $\phi_i \wedge \phi_j$ is not satisfiable and $\phi \wedge \phi_i$ is satisfiable (for $i, j \in \{1, ..., m\}$ and $i \neq j$), m is maximum, $\psi \circ (\mu \vee \phi) \equiv \phi$ and $\psi \circ (\mu \vee \phi_1 \vee ... \vee \phi_{i-1} \vee \phi_{i+1} \vee ... \vee \phi_m) \equiv (\mu \vee \phi_1 \vee ... \vee \phi_{i-1} \vee \phi_{i+1} \vee ... \vee \phi_m)$. Let $\alpha \equiv (\alpha_1 \vee ... \vee \alpha_k)$ such that $\alpha_i \wedge \alpha_j$ is not satisfiable and $\alpha \wedge \alpha_i$ is satisfiable (for $i, j \in \{1, ..., k\}$ and $i \neq j$), k is maximum, $\psi \circ (\mu \vee \alpha) \equiv \alpha$ and $\psi \circ (\mu \vee \alpha_1 \vee ... \vee \alpha_{i-1} \vee \alpha_{i+1} \vee ... \vee \alpha_k) \equiv (\mu \vee \alpha_1 \vee ... \alpha_{i-1} \vee \alpha_{i+1} \vee ... \vee \alpha_k)$. If $\phi \wedge \alpha$ is not satisfiable and $\min(m, k) \geqslant 2$ as well as $m * k < n$ then there exist $\emptyset \neq I \subsetneq \{1, ..., m\}$ and $\emptyset \neq J \subsetneq \{1, ..., k\}$ such that $\psi \circ (\mu \bigvee_{i \in I} \phi_i \bigvee_{j \in J} \alpha_j) \equiv (\bigvee_{i \in I} \phi_i \bigvee_{j \in J} \alpha_j)$

While we have some understanding of interdependence between two disjoint blocking sets for some $\omega \in \Omega$ in weak faithful cyclic orders, we do not have the same for a random number of disjoint blocking sets. The following example shows a possible cyclic order that entails exactly three disjoint blocking sets:

Example 13. *Let* $\mathcal{N}_1 = \{\omega_a, \omega_b\}$, $\mathcal{N}_2 = \{\omega_1, \omega_2\}$, $\mathcal{N}_3 = \{\omega_I, \omega_{II}\}$, $\mathcal{M}_\psi = \{\omega_{\psi_1}, \omega_{\psi_2}, \omega_{\psi_3}, \omega_{\psi_4}, \omega_{\psi_5}, \omega_{\psi_6}, \omega_{\psi_7}, \omega_{\psi_8}\}$ *and*

$$\begin{gathered}\mathcal{C}_\psi = \{\\ (\omega_{\psi_1}, \omega_a, \omega), (\omega_{\psi_2}, \omega_a, \omega), (\omega_{\psi_3}, \omega_a, \omega), (\omega_{\psi_4}, \omega_a, \omega),\\ (\omega_{\psi_5}, \omega_b, \omega), (\omega_{\psi_6}, \omega_b, \omega), (\omega_{\psi_7}, \omega_b, \omega), (\omega_{\psi_8}, \omega_b, \omega),\\ (\omega_{\psi_1}, \omega_1, \omega), (\omega_{\psi_3}, \omega_1, \omega), (\omega_{\psi_5}, \omega_1, \omega), (\omega_{\psi_7}, \omega_1, \omega),\\ (\omega_{\psi_2}, \omega_2, \omega), (\omega_{\psi_4}, \omega_2, \omega), (\omega_{\psi_6}, \omega_2, \omega), (\omega_{\psi_8}, \omega_2, \omega),\\ (\omega_{\psi_1}, \omega_I, \omega), (\omega_{\psi_2}, \omega_I, \omega), (\omega_{\psi_5}, \omega_I, \omega), (\omega_{\psi_6}, \omega_I, \omega),\\ (\omega_{\psi_3}, \omega_{II}; \omega), (\omega_{\psi_4}, \omega_{II}; \omega), (\omega_{\psi_7}, \omega_{II}; \omega), (\omega_{\psi_8}, \omega_{II}; \omega)\}.\end{gathered}$$

We see in table 4 that there is no additional $\mathcal{N}' \subsetneq \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ *such that* $\mathcal{N}' \neq \mathcal{N}_i$ *for* $i \in \{1, 2, 3\}$.

A reasonable assumption is that multiple disjoint blocking sets $\mathcal{N}_1, ..., \mathcal{N}_n$ are possible if $(\prod_{1 \leqslant i \leqslant n} |\mathcal{N}_i|) \leqslant |\mathcal{M}_\psi|$.

Another reasonable assumption is that if we have a set of blocking sets $\mathcal{N}_1, ..., \mathcal{N}_n$ that are not disjoint, i.e. there exists $J \subseteq I = \{1, ..., n\}$ and such that $|J| > 1$ and $\mathcal{N}^J = \bigcap_{j \in J} \mathcal{N}_j$ is non empty, then $|\mathcal{N}^J| * \prod_{i \in I} |\mathcal{N}_i \backslash \mathcal{N}^J| \leqslant |\mathcal{M}_\psi|$.

In any case the further study of blocking sets is very likely necessary to formulate a representation theorem for weak faithful cyclic orders.

5 Conclusion

In the course of this thesis we formulated and proofed a representation theorem for a certain class of cyclic orders. For this theorem we defined a concept of closeness and with that of minimal change in cyclic orders, similar to the versions of Katsuno and Mendelzon for binary orders. We showed, that in order to satisfy postulate (R3), we need a less strict concept of closeness for cyclic orders, than for binary orders. Because our concept of closeness in cyclic orders is more lenient than the one for binary orders, there are cyclic orders that do not satisfy postulate (R8). That left us with two options: to either narrow down a class of cyclic orders that do satisfy postulate (R8) or to find additional postulates that characterise all cyclic orders. We found a condition that if fulfilled by a cyclic order, guarantees the satisfaction of postulate (R8). With the definition of strong faithful assignments we were able to formulate a representation theorem for strong faithful cyclic orders. Because strong faithful cyclic orders satisfy the same postulates as partil (pre-)orers, we were able to make use of the relation of partial orders and partial cyclic orders. Hence we adopted the construction for faithful partial (pre-)orders, proposed by Katsuno and Mendelzon, in order to construct strong faithful cyclic orders. With this we proofed the representation theorem for strong faithful cyclic orders.

In the case of weak faithful cyclic orders we investigated the condition which contradicts postulate (R8). We found a definition and gave it the name 'blocking sets'. A blocking set $\mathcal{N} \subseteq \Omega$ for some possible world ω is a minimal set such that $\mathrm{Mod}(\psi \circ \mathrm{form}(\mathcal{N}, \omega)) = \mathcal{N}$, while minimal means that for all $\mathcal{N}' \subsetneq \mathcal{N} : \mathrm{Mod}(\psi \circ \mathrm{form}(\mathcal{N}', \omega)) = \mathcal{N}' \cup \{\omega\}$. After formulating a group of simple postulates (C1a), (C1b), (C1c) and a postulate to ensure transitivity of blocking sets (C2), we showed that the size of blocking sets in weak faithful cyclic orders is limited by the size of $\mathrm{Mod}(\psi)$ (or the size of a set $\mathcal{M}_\psi \neq \emptyset$ if $\mathrm{Mod}(\psi) = \emptyset$). As a consequence we formulated postulate (C3).

We showed that in order to find a representation theorem for weak faithful cyclic orders, a weak faithful cyclic assignment must be defined. For this purpose we investigated how multiple blocking sets for some ω, interfere with each other. We found out that if a revision has two blocking sets for the same ω such that there is no other blocking set for ω, then the product of their respective size is at most the size of $\mathrm{Mod}(\psi)$. As a consequence we formulated postulate (C4). It is likely that further investigation on the interdependence of multiple blocking sets and a postulate summarizing these results could replace both (C3) and (C4). In the end our results were not enough to define a weak faithful assignment for a representation theorem for weak faithful cyclic orders.

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