Continuous-Discrete Filtering using the Zakai Equation: Smooth Likelihood Surface

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Continuous Time State Space Model

$$
\begin{aligned}
dY(t) &= f(Y(t), t)dt + G(Y(t), t)dW(t) \\
dZ(t) &= h(Y(t), t)dt + dV(t)
\end{aligned}
$$

sampled measurements:

$$
z_i = h(Y(t_i), t_i) + \epsilon_i
$$

- Goal: Optimal Filtering and Maximum Likelihood Estimation
- Wiener process $W(t)$
- Itô stochastic differential equations
- sampled measurements $\dot{Z}(t_i) = z_i$, $\rho(t)/dt = R(t)$

(some) Solution Methods

- Kalman Filtering (sequential)
  - Moment based methods
    - Taylor expansion: EKF, SNF, HNF
    - Numerical integration: UKF, GHF, Smolyak sparse grid
  - PDE based methods: Stratonovich-Kushner and Duncan-Mortensen-Zakai (DMZ) equation
  - Exact filters: Daum, Benes
  - Particle Filters:
    Sequential Monte Carlo
- Markov Chain Monte Carlo (nonsequential)
  - Simulated likelihood
  - Bayesian approaches

Nonlinear bistable diffusion: Ginzburg-Landau model

$$
\begin{aligned}
dY &= -[\alpha Y + \beta Y^3]dt + \sigma dW(t) \\
&= -\nabla \Phi(Y) + \sigma dW(t) \\
z_i &= Y(t_i) + \epsilon_i
\end{aligned}
$$

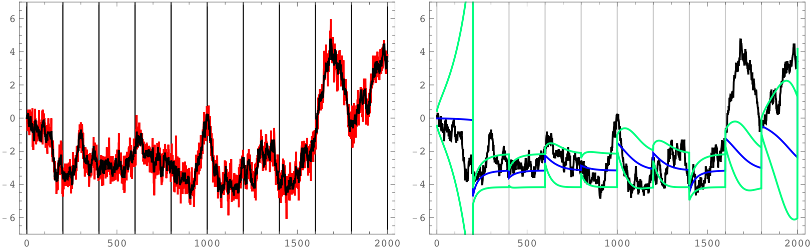

Figure: Simulated data (left) and extended Kalman filter (right).

Likelihood surface: Particle filter and Gauß-Hermite filter

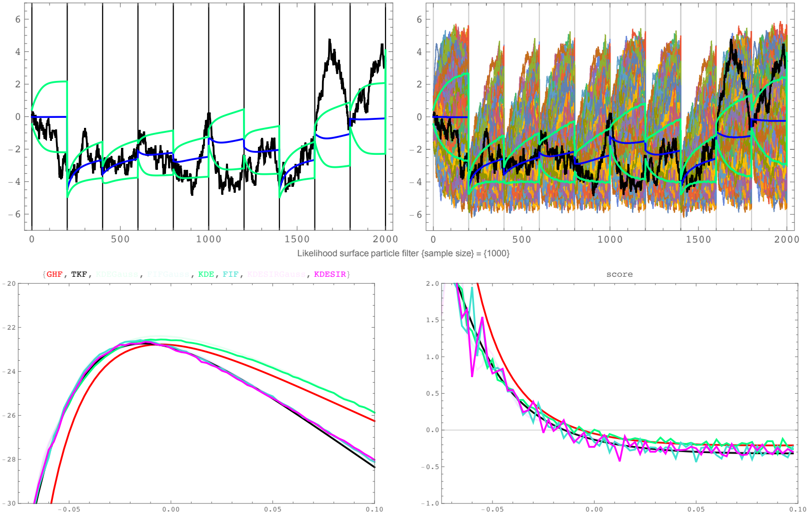

Figure: SIR particle filter (mean and SD) and trajectories (top, right), likelihood and score for β (bottom). Increment $d\beta = 0.0025$.

Economic example: Equilibrium model: Herings (1996)

Potential $\Phi(y) \sim \frac{\alpha}{2}y^2 + \frac{\beta}{4}y^4$

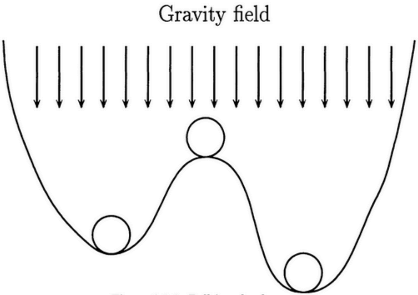

Figure 1.1.1. Ball in a landscape.

science, it is difficult to resist making a comparison between the states of an economy and the states of a ball in a landscape (see Figure 1.1.1 to clarify some concepts).

State estimation: Continuous-discrete Kalman Filter

time update: $t_i \leq t < t_{i+1}$ (Fokker-Planck equation)

$$\partial_t p(y, t|Z^i) \quad = \quad F(y, t)p(y, t|Z^i)$$

measurement update: $t = t_{i+1}$ (Bayes formula)

$$p(y_{i+1}, t_{i+1}|z_{i+1}, Z^i) \quad = \quad \frac{p(z_{i+1}, t_{i+1}|y_{i+1}, Z^i)p(y_{i+1}, t_{i+1}|Z^i)}{p(z_{i+1}, t_{i+1}|Z^i)}$$

- filter density $p(y, t|Z^t)$
- Fokker-Planck operator $F(y, t) = -\partial_\alpha f_\alpha + \frac{1}{2}\partial_\alpha \partial_\beta \Omega_{\alpha\beta}$

Recursive likelihood

$$
\begin{aligned}
p(z_{i+1}|Z^i;\psi) &= \int p(z_{i+1}|y_{i+1},Z^i)p(y_{i+1}|Z^i)dy_{i+1} \\
&:= \int u(y_{i+1}|z_{i+1},Z^i)dy_{i+1} \\
&\approx \sum_l w_l \; u(y_{i+1,l}|z_{i+1},Z^i)
\end{aligned}
$$

- unnormalized filter density $u(y,t|Z^t)$
- numerical integration using quadrature formulas
- measurements up to time t_i: $Z^i = \{Z(s) \mid s \leq t_i\}$

Continuous time filtering: DMZ equation

SPDE: Zakai (1969)

$$
\begin{aligned}
\partial_t u(y,t|Z^t) &= [F + h'\rho^{-1}(\dot{Z} - h/2)] \circ u(y,t|Z^t) \\
&= [F(y,t) + M(y,t)] \circ u(y,t|Z^t)
\end{aligned}
$$

- measurement precision $\rho^{-1}(y,t) = \sum_i \pi(t-t_i)(R_i dt)^{-1}$
- measurement density

  $$p(dZ(t)|y,Z^t) \propto \exp\left\{-\frac{1}{2}(dZ - hdt)'(\rho dt)^{-1}(dZ - hdt)\right\}$$

- $dZ \circ u$: symmetrized product

Stochastic Representation: Feynman-Kac Formula

$$u(y, t|Z^t) \;=\; E\left[e^{\int_{t_0}^{t} M(Y(\tau),\tau)d\tau}\delta(y - Y(t)) \,\Big|\, Z^t\right]$$

use Lie -Trotter and Zassenhaus formula

$$e^{(F+M)\delta t}\delta(y - y') \approx e^{M\delta t}e^{F\delta t}\delta(y - y') = e^{M\delta t}p(y, t + \delta t|y', t)$$

- $dY(t) = f(Y, t)dt + G(Y, t)dW(t),\; Y(t_0) \sim p(y, t_0|Z^{t_0})$
- Dirac delta function $\lim_{n\to\infty} \int \delta_n(x)\phi(x)dx = \phi(0)$

Importance Sampling: Backward DMZ Equation

time reversal $c(x,s) = u(x, T-s), s \leq T$

$$\begin{aligned} \partial_s c + Lc + (M+v)c &= 0 \\ \text{terminal condition } c(x,T) &= h(x) = u(x,0) \end{aligned}$$

$$c(x,s) = E\left[e^{\int_s^T (M+v)(X(\tau),\tau)d\tau} h(X(T)) \,\middle|\, X(s) = x, Z^{T-s} \right]$$

- $dX(\tau) = \tilde{f}(X, T-\tau)dt + G(X, T-\tau)dW(\tau),\ X(s) = x$
- backward operator $L = [-f_\alpha + (\partial_\beta \Omega_{\alpha\beta})]\partial_\alpha + \frac{1}{2}\Omega_{\alpha\beta}\partial_\alpha\partial_\beta$
- scalar potential $v = -(\partial_\alpha f_\alpha) + \frac{1}{2}(\partial_\alpha\partial_\beta\Omega_{\alpha\beta})$

Simulation of Backward DMZ Equation

Stochastic representation

$$\begin{aligned} c(x,s) &= E\left[e^{\int_s^T (M+v)(X(\tau),\tau)d\tau} h(X(T)) \,\middle|\, X(s)=x \right] \\ dX(\tau) &= \tilde{f}(X, T-\tau)dt + G(X, T-\tau)dW(\tau) \\ X(s) &= x \end{aligned}$$

- Importance sampling: drift correction (Milstein; 1995)

  $\Omega(X, T-\tau)\nabla \log u(X, T-\tau)$

- approximate filter solution $\hat{u}(X, T-\tau)$:
- EKF, GHF, UKF or particle filter

Ginzburg-Landau Model: Forward and backward simulation

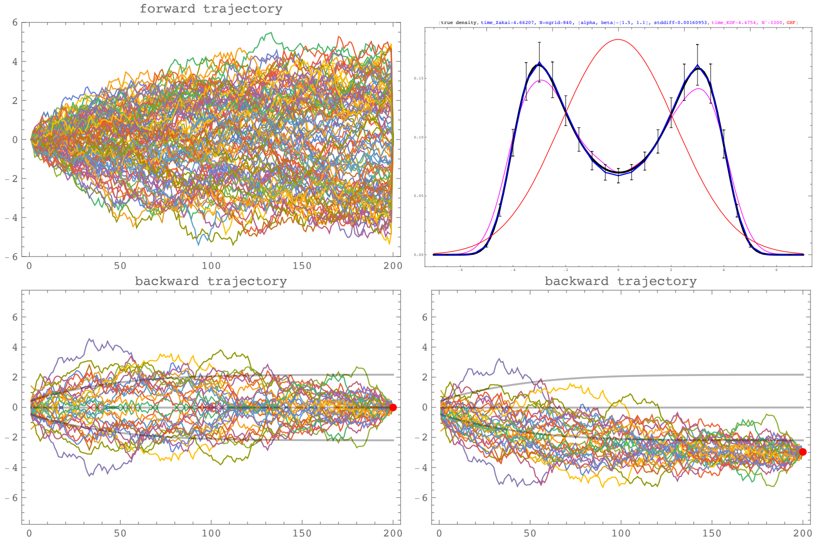

Figure: Estimated filter density (top, right), backward simulation (bottom) and forward simulation (top, left)

Ginzburg-Landau Model: Forward and backward simulation

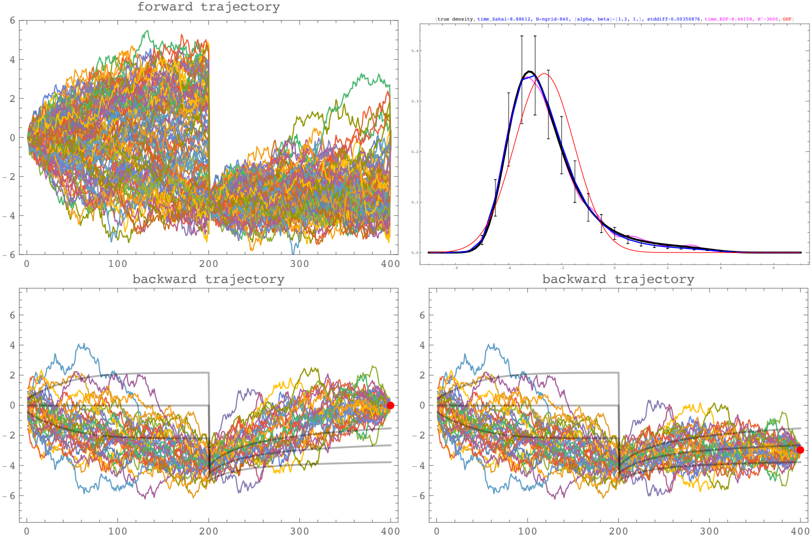

Figure: Estimated filter density (top, right), backward simulation (bottom) and forward simulation (top, left)

Likelihood surface: Particle filter and Zakai Equation

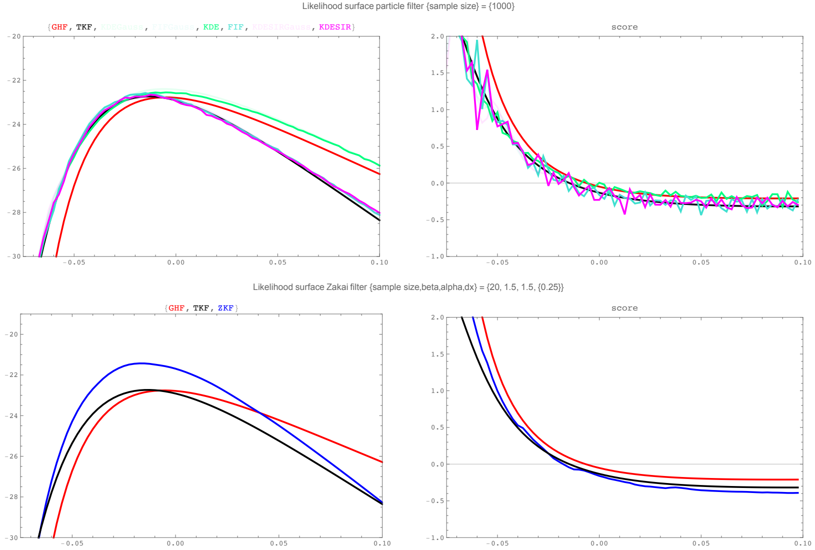

Figure: Likelihood for SIR particle filter (top) and ZKF (Riemann), GHF, TKF (Riemann). Increment $d\beta = 0.0025$.

Likelihood surface: Zakai Equation (UT)

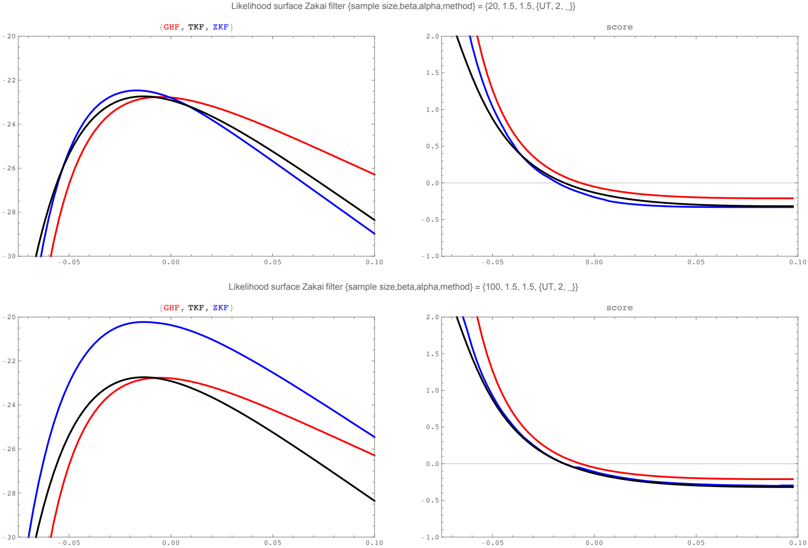

Figure: Likelihood for ZKF (unscented transform UT), GHF, TKF.
$N = 20, 100$. Increment $d\beta = 0.0025$.

Conclusions

- Continuous-discrete filtering
  with continuous time measurement equation
- Feynman-Kac representation
  of backward Zakai equation
- Variance reduced simulation
  of unnormalized filter density
  at supporting points
- No resampling required
- Smooth likelihood approximation
  using quadrature formulas at supporting points

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Operator splitting

Lie –Trotter formula

$$\lim_{n\to\infty}[e^{At/n}e^{Bt/n}]^n \quad = \quad e^{(A+B)t}$$

Zassenhaus formula

$$\begin{aligned} e^{\lambda(A+B)} &= e^{\lambda A}e^{\lambda B}e^{\lambda^2 C_2}e^{\lambda^3 C_3}\ldots. \\ C_2 &= \tfrac{1}{2}[B, A] \\ C_3 &= \tfrac{1}{3}[C_2, A+2B] \end{aligned}$$

$$e^{(A+B)} \quad \approx \quad \left[e^{A/n}e^{B/n}e^{C_2/n^2}e^{C_3/n^3}\ldots e^{C_m/n^m}\right]^n$$

Stratonovich calculus

$$dZ(t)u(y,t) \quad = \quad dZ(t) \circ u(y,t) - \tfrac{1}{2}h(y,t)u(y,t)dt$$

DMZ equation in Itô-form

$$du(y,t|Z^t) \quad = \quad [F(y,t)dt + h'(y,t)\rho^{-1}(t)dZ(t)]u(y,t|Z^t)$$

symmetrized product

$$\begin{aligned} dZ(t) \circ u(y,t) &:= dZ(t)\bar{u}(y,t) \\ \bar{u}(y,t) &:= \tfrac{1}{2}[u(y,t) + u(y,t+dt)] \\ u(y,t) &= \bar{u}(y,t) - \tfrac{1}{2}du(y,t) \end{aligned}$$

Potential $\Phi(y) = \frac{\alpha}{2}y^2 + \frac{\beta}{4}y^4$, drift $f(y) = -\nabla\Phi$

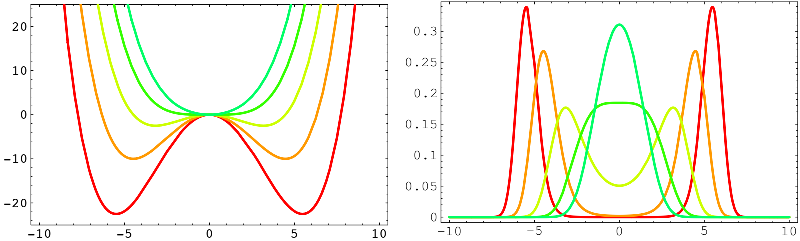

Figure: Left: Potential as a function of y for parameter values $\alpha = -3, -2, ..., 1$. Right: Stationary density $p_{stat} \propto \exp[-(2/\sigma^2)\Phi(y)]$.

Importance sampling: Kolmogorov Backward Equation

$$\begin{aligned} \partial_s c(x,s) + L(x,s)c(x,s) + v(x,s)c(x,s) &= 0 \\ \text{terminal condition } c(x,T) &= h(x) \end{aligned}$$

solution

$$c(x,s) = E\left[e^{\int_s^T v(Y(\tau),\tau)d\tau} h(X(T)) \,\middle|\, X(s) = x \right]$$

- $dX(t) = f(X,t)dt + G(X,t)dW(t)$, $X(s) = x$
- importance sampling: drift correction $\Omega(x,s)\nabla \log c(x,s)$ (Milstein; 1995)
- backward operator $L = f_\alpha \partial_\alpha + \frac{1}{2}\Omega_{\alpha\beta}\partial_\alpha\partial_\beta$