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ORIENTED MATROIDS FROM WILD SPHERES

W. HOCHSTÄTTLER
Institut für Mathematik
BTU-Cottbus, Deutschland
e-mail: hochst@math.tu-cottbus.de

In this paper we give a new proof of one direction of the Topological Representation Theorem for Oriented Matroids. Our proof does not impose any "niceness" condition on the hyperspheres of the representing system, and thus generalizes the result in the strongest possible way. This solves a problem presented by A. Björner (see [1] Exercise 5.6.c).

Introduction

Matroids, introduced in the 30s of this century, were modelled as a combinatorial abstraction of linear dependency in a vector space. Though they, originally, were supposed to reflect a geometric situation their axioms are weak enough to allow structures quite different from point configurations in Euclidean space. As a consequence they are frequently considered to be "just" set systems.

When oriented matroids were introduced in [8], [2] the former of these articles proved a close connection to topology: Every oriented matroid can be realized as a cell decomposition of a sphere induced by codimension 1 spheres such that the signed covectors of the oriented matroid correspond bijectively to the cells of the complex. Thus, oriented matroids always encode a geometric situation.

On the other hand certain systems of hyperspheres on a sphere were shown to induce oriented matroids (see [8] and [10]). In both cases a "niceness" condition was imposed on the hyperspheres, namely that their embedding always can be extended to a global homeomorphism of the sphere. As the hyperspheres are nice, they induce a regular cell decomposition of the sphere, which is the cell complex realizing the poset of the signed covectors of an oriented matroid.

Recently, an elementary proof of the rank 3 case of the Folkman-Lawrence Theorem has been given by Bokowski, Mock and Streinu [3]. It is well known that any topological hypersphere of the 2-sphere is "nice" and thus, no extra niceness condition needs to be imposed on the hyperspheres. This is different in higher dimension where topologically "wild" hyperspheres do exist that do not seperate the sphere into two open balls. By Jordan Brouwer Separation Theorem arbitrarily wild spheres have the combinatorial property, though, to separate the sphere, thus, there does not seem to be a combinatorial reason, why the topological property of "being nice" should be necessary for a collection of hyperspheres to encode the combinatorial structure of an oriented matroid.

In this paper we will show that even without any niceness condition a sphere system will always give rise to the set of signed vectors of an oriented matroid. As a corollary we derive an even weaker definition of a sphere system. Finally, we will restate the results in a lattice theory language to derive a characterization of oriented matroids in terms of shellability. We will prove that oriented matroid posets are given as matroidal nested systems of spheres satisfying a certain symmetry property.

The paper is organized as follows: In Section 1 we will introduce sphere systems and some tools, in Section 2 we will present our main result and its proof. Section 3 proves that the axiom system for sphere systems can be weakened without loosing structure and in Section 4 we will present a lattice theoretical version of that result. We assume some familiarity with matroids and oriented matroids, standard references are [12] and [1]. Furthermore we will need some basic facts in elementary topology on an undergraduate course level (see e. g. [9]).

1. Basics and definitions

Why is it natural to consider sphere systems in the context of oriented matroids? Oriented matroids may be viewed as a combinatorial abstraction of affine hyperplane arrangements. The difficulty with boundary effects of parallel flats as usual can be overcome by adding the hyperplane at infinity, thus considering the situation projectively.

One common definition of the projective space constructs it from the standard sphere by identifying antipodal points. The resulting space is not orientable which conflicts intuition. We can avoid this by not doing the identification and considering the sphere as two-sided oriented projective space with the side effect that vector spaces become spheres. That way hyperplane arrangements "become hypersphere arrangements".

Definition 1. *Let* $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ *denote the n-dimensional standard sphere. If* $X \subset S^n$ *is homeomorphic to* S^{n-1}*, we call* X *a* hypersphere *of* S^n.

The most important property of a hypersphere on a sphere for our purposes is given by the Jordan-Brouwer Separation Theorem (see e. g. [7]).

Theorem 1 (Jordan-Brouwer Separation Theorem). *Any hypersphere* H_e *of* S^n *separates* S^n *into exactly two disjoint domains of which it is the common boundary.*

Although combinatorially any hypersphere is, therefore, well-behaved, the components may have non-trivial topology (such spheres are called *wild*).

Example 1. *A first example of a sphere wildly embedded into* $\mathbb{R}^3$ *was given by Alexander (see e. g. [11]). The construction is done by the following iterative process.* S_0 *is a standard sphere, this sphere grows two horns which almost link to get* S_1*, each of these horn splits into two fingers which almost link and the pairs of fingers mesh thus forming* S_2*. Iterating this process we get uniform convergence towards a homeomorphic image* S_∞ *of the sphere. The unbounded component of* $\mathbb{R}^3 \setminus S_\infty$ *has non-trivial topology, it is not simply connected. A sketch of* S_3 *is given in Fig. 1.*

For notational reasons it will be convenient given a hypersphere of S^n to mark one of the two components of its complement.

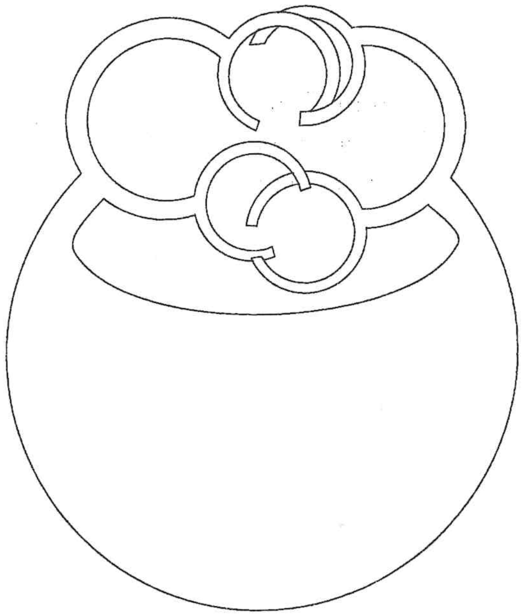

Fig. 1. The third step on the way to Alexander's horned sphere.

Definition 2. *Let H_e be a hypersphere of S^n. Marking one of the two components of $S^n \setminus H_e$ to be the* positive side H_e^+ *we get an* oriented hypersphere. *Let $H_e^- := S^n \setminus (H_e \cup H_e^+)$. We will use H_e^0 and H_e synonymously.*

Let $\mathcal{H} = (H_e)_{e \in E}$ be a family of oriented hyperspheres of S^n. If $A \subseteq E$, we call $\bigcap_{f \in A} H_f$ the flat *of A. Furthermore we denote by $cl(A) = \{e \in E \mid H_e \supseteq \bigcap_{f \in A} H_f\}$ the* closure *of A.*

$\mathcal{H}$ is called a sphere system *if*

(S1) All flats are topological spheres.

(S2) The intersection of a flat F with any oriented hypersphere $H_e \not\supseteq F$, is a hypersphere $H_e \cap F$ of F, separating F into $F^+ \subseteq H_e^+$ and $F^- \subseteq H_e^-$.

If $s : E \to \{+, -, 0\}$ is some map then we call

$$\bigcap_{e \in E} H_e^{s(e)}$$

a pseudo cell.

Remark 1. *Note, that a pseudo cell is not necessarily homeomorphic to a ball. The author does not even know whether it is always connected.*

Considering only hyperspheres which are tamely embedded, i. e. the two components of their complements are simply connected, the above axioms are known to define a regular CW-complex the poset of which is isomorphic to the big face lattice of an oriented matroid (see [1]). Take a hypersphere H_f of such a tame sphere system and consider a cell Z of $H_f \setminus \bigcup_{e \in E \setminus f} H_e$. This cell will be in the boundary of two maximal cells T_1 and T_2. Now observe that given a cell decomposition of a standard S^n-sphere embedded in $\mathbb{R}^n$ the modifications of the sphere to get Alexander's horned sphere can be restricted to any maximal cell of that decomposition. Therefore, it is clear that we can have H_f grow horns such that say T_2 is not simply connected *anymore but neither the topology of any other cell changes nor any boundary relation changes. Nevertheless, it is not very surprising that we get a "pseudo cell complex" with the same poset as before. This will be made more precise later (see Definition 4).*

In other words it is immediate that there are oriented matroids "from" wild spheres. What we are going to prove in the following is that the above axiom system is strong enough to always guarantee an oriented matroid.

Two minors of sphere systems are immediate.

Lemma 1. *Let $\mathcal{H}$ be a sphere system on a finite set $E, A \subseteq E, G = \bigcap_{e \in A} H_e$. Then*

1. $\mathcal{H}_{|A} := \{H_e \in \mathcal{H} \mid e \in A\}$,

2. $\mathcal{H}/A := \{f(H_e \cap G) \mid e \in E \setminus A\}$, *where $f : G \to S^d$ is a suitable homeomorphism*

are sphere systems.

To prove that the pseudo cells of any sphere system give rise to the signed covectors of an oriented matroid we are going to use our result from [6]. There we showed that a map from a lattice to a geometric lattice is the zero map of an oriented matroid if and only if it is a symmetric antitonic folding. The result is quite technical. Therefore we cannot avoid making the following definitions.

Definition 3. *Let $\mathcal{P}$ be a finite partially ordered set. We say $\mathcal{P}$ is* connected, *if for any two elements $a, b \in \mathcal{P}$ there is a path in the Hasse-Diagram from a to b. We say $\mathcal{P}$ is* openly connected, *if all open intervals with at least two comparable elements are connected. If $I = [x, y]$ is an interval in a poset, the* length of I *is given by the number of elements in a maximum chain minus one. $\mathcal{P}$ has the* diamond property, *if each interval of length 2 consists of exactly four elements.*

A finite lattice $\mathcal{R}$ is a combinatorial d-manifold *[4], if*

- *$\mathcal{R}$ is openly connected,*
- *$\mathcal{R}$ has length $d + 2$,*
- *$\mathcal{R}$ has the diamond property.*

If $\mathcal{P}$ is a poset and $a < b \in \mathcal{P}$, then b covers a, *if for all $c \in \mathcal{P}$ we have $a \leq c \leq b$ only if $a = c$ or $b = c$. We denote this by $a \lessdot b$ or $b \gtrdot a$. If $\mathcal{P}$ und $\mathcal{R}$ are JD-lattices of finite length and $r(\mathbb{1}_\mathcal{P}) = r(\mathbb{1}_\mathcal{R}) + 1$ then an* antitonism *is a map $\varphi : \mathcal{P} \setminus \{\mathbb{1}_\mathcal{P}\} \to \mathcal{R}$ satisfying*

- $(a \lessdot b \in \mathcal{P}$ *and* $b \neq \mathbb{1}_\mathcal{P}) \Longrightarrow \varphi(a) \gtrdot \varphi(b)$,
- $(a, b \in \mathcal{P}$ *and* $a \vee b \neq \mathbb{1}_\mathcal{P}) \Longrightarrow \varphi(a \vee b) = \varphi(a) \wedge \varphi(b)$.

Let $\mathcal{P}$ be a combinatorial d-manifold and L be a geometric lattice of rank $d+1$. An antitonic folding *φ is a surjective antitonism $\varphi : \mathcal{P} \setminus \{\mathbb{1}_\mathcal{P}\} \to L$ satisfying*

folding property: *If $[x, y] \subseteq L$ is an interval of length 1 and $b \in \varphi^{-1}(y)$, then there are exactly two elements $a, a' \in \mathcal{P}$ in the preimage of x covering b. In this situation we call the triple (a, b, a') a* fold.

local flatness: *If* $[x, y] \subseteq L$ *is an interval of length* 2 *and* $b \in \varphi^{-1}(y)$, *then the Hasse-Diagram of the set* $\{a \in \mathcal{P} \mid a > b \text{ und } \varphi(a) \geq x\}$ *is connected.*

We say an antitonic folding $(L, \mathcal{P}, \varphi)$ *is* symmetric, *if for all folds* $a, 0, a'$ *and any path* $a = a_0, a_1, \ldots, a_n = a'$ *in* $\mathcal{P} \setminus \{\mathbb{1}_\mathcal{P}\}$ *we have:* $\forall 0 \lessdot p \in L\, \exists 0 \leq i \leq n : p \leq \varphi(a_i)$.

Theorem 2 ([6]). *Let* $(L, \mathcal{P}, \varphi)$ *be a symmetric antitonic folding. Then* $\mathcal{P}$ *is the big face lattice of an oriented matroid with* φ *for its zero-map.*

2. Oriented matroids from wild spheres

While Theorem 2 actually characterizes reorientation classes of oriented matroids we could actually examine sphere systems without specified orientations. But since we provided each sphere in our system with an orientation the Separation Theorem of Jordan Brouwer immediately yields a unique signed vector $X \in 2^{\pm E}$ for each point $p \in S^n$ und thus for each pseudo cell, rendering notation easier.

Definition 4. *A* signed vector U *on a finite set* E *is a map* $U : E \to \{0, +, -\}$. *Instead of* $U(e)$ *we will simply write* U_e. *If* U *and* V *are signed vectors* $sep(U, V) := \{e \in E \mid U_e = -V_e \neq 0\}$ *and* $supp(U) := U^{-1}(\{+, -\})$.

On signed vectors we have the partial order $\mathcal{P}$:

$$X \preceq Y :\Leftrightarrow \forall e \in E : (X_e \neq Y_e \Rightarrow X_e = 0).$$

In the following we will prove that a family $\mathcal{H}$ of oriented hyperspheres induces a symmetric antitonic folding in a natural way.

We have a canonical right hand side for an antitonism:

Lemma 2. *The map cl satisfies the axioms of a matroid closure.*

Proof. By set inclusion we get an atomic finite lattice on the image of $\mathrm{cl} : 2^E \to 2^E$ with the lattice operations $\sup\{A, B\} = \mathrm{cl}(A \cup B)$ und $\inf\{A, B\} = A \cap B = \mathrm{cl}(A \cap B)$ for two sets A, B in the image of cl. The first half of Axiom $(S2)$ implies that in this lattice L the following holds: If $e \in L$ is atomic and $x \in L$ such that $e \not\leq x$, then $e \vee x \gtrdot x$. Hence Theorem 3 implies that L is geometric. The lemma follows by the 1-1-correspondence between geometric lattices and simple matroids (see [12] 3.4.1). ∎

Theorem 3 (see e. g. [12] Exercise 3.8). *A finite, atomic lattice is geometric if and only if*

$$\forall a \in L \quad \forall 0 \lessdot p \in L : \qquad (p \not\leq a \Rightarrow a \lessdot a \vee p).$$

Thus cl provides us with the possibility to construct a map onto a geometric lattice. Before we give its definition we mention the following fact, the easy verification of which is left to the reader.

Proposition 1. *Let* $\mathcal{H}$ *be a sphere system and* X *one of its signed vectors. Then* $cl(E \setminus supp(X)) = E \setminus supp(X)$.

Definition 5. *For all $X \in \mathcal{P}$ let $\varphi(X) := E \setminus supp(X)$.*

By the above proposition φ is a map from some poset onto a geometric lattice. With the next lemma we prove that $\mathcal{P}$ is pure dimensional.

Lemma 3. *Let $\mathcal{P}$ be the poset of a sphere system. Then:*

1. Each point of the pseudo cell $0_{\mathcal{P}}$ is in the boundary of every component of $S^n \setminus \bigcup_{e\in E} H_e$.

2. If $x \in S^n$ with signed vector X, then for all $Y \in \mathcal{P}$:

$$X \prec Y \Leftrightarrow x \text{ is in the boundary of } Y.$$

3. $\mathcal{P}$ satisfies the JORDAN-DEDEKIND *chain condition.*

4. $\mathcal{P}$ is pure dimensional.

Proof.

1. We proceed by induction on $m = |E|$. The case $m = 1$ is clear by Jordan Brouwer Separation Theorem. Let $m \geq 2$, and $p \in \bigcap_{e\in E} H_e$, let Θ be a component of $S^n \setminus (\bigcup_{e\in E} H_e)$ and $f \in E$. In the sphere system $(H_e)_{e\in E\setminus f}$ there is a component Θ' of $S^n \setminus \left(\bigcup_{e\in E\setminus f} H_e\right)$ such that $\Theta \subseteq \Theta'$. If $\Theta' = \Theta$ we are done. Otherwise the intersection of the boundary of Θ with H_f is non-empty and thus contains a component θ of $H_f \setminus \left(\bigcup_{e\in E\setminus f} H_e\right)$. By inductive assumption p is in the boundary of θ. Since θ is contained all in the boundary of Θ, the claim follows.

2. and 3. The backward implication of 2. is obvious. Considering the statement of 3. we first remark that each pseudo cell different from $0_{\mathcal{P}}$ is a relatively open proper subset of the minimal flat containing it. Hence the dimension of a pseudo cell is well defined. Instead of 3. we may thus prove that $X \preceq Y$ and Y covers X then $\dim(X) = \dim(Y) - 1$.

   To verify sufficiency of 2. and to prove 3. we proceed by induction on $m = |\text{supp}\,(X)|$. The case $m = 0$ is clear from the first claim of this lemma. Thus let $f \in \text{supp}\,(X) \neq \emptyset$. Let X', Y' denote the pseudo cells of $(H_e)_{e\in E\setminus f}$, containing X resp. Y. Necessarily we must have $\dim(X') = \dim(X)$ and $\dim(Y') = \dim(Y)$. If Y' still covers X' we are done (for all $x \in X$ we have some sufficiently small neighborhood intersecting Y' only in points of Y). Therefore, assume there is some Z' such that $X' \prec Z' \prec Y'$. Since Y covers X no point of Z' must be on the same side of H_f than any point of X, contradicting the fact that each point of X must be in the boundary of Z'.

   Note, that in fact we proved that the rank function of $\mathcal{P}$ satisfies $\forall X \in \mathcal{P} : r(X) = \dim(X) + \dim(0_{\mathcal{P}})$.

4. This is immediate from the above and the fact that each point which is not an interior point of a full dimensional pseudo cell is in the boundary of some full dimensional pseudo cell.

■

The following is a direct consequence of the definition of a sphere system.

Lemma 4. *Let $\mathcal{H} = (H_e)_{e \in E}$ be a sphere system, X a signed vector of the corresponding partial order $\mathcal{P}$ and $\varphi(X) = A$. The partial order of $\mathcal{H}/A$ is isomorphic to the order ideal generated by $\varphi^{-1}(A)$.*

The other minor is also compatible with the partial order.

Lemma 5. *Let $\mathcal{H} = (H_e)_{e \in E}$ be a sphere system, X one of its signed vectors and $A = \varphi(X)$. The partial order of $\mathcal{H}_{|A}$ is isomorphic to the filter generated by X in $\mathcal{P}$.*

Proof. It suffices to verify that V is a signed vector of $\mathcal{H}_{|A}$ if and only if (X, V) is a signed vector of $\mathcal{P}$ (where (X, V) denotes the signed vector satisfying $(X, V)_{|\text{supp}\,(X)} = X_{|\text{supp}\,(X)}$ and $(X, V)_{|A} = V$).

The non-trivial implication is sufficiency. This is done by induction on $n = |E|$. The case $n = 0$ is clear, thus let $n > 0$. If $\varphi(V_{|A}) \neq \emptyset$, then all points with vectors V or X are contained in the sphere $\bigcap_{e \in \varphi(V_{|A})} H_e$, and the claim follows from inductive hypothesis and Lemma 4. So we may assume $\varphi(V_{|A}) = \emptyset$ and $V \neq 0$ and choose a signed vector W of $\mathcal{H}_{|A}$ such that $W \prec V$ and V covers W. Let $f \in \text{supp}\,(V) \setminus \text{supp}\,(W)$. By inductive hypothesis applied to $\mathcal{H}/f$ and Lemma 4, (X, W) is a signed vector of $\mathcal{P}$. By Jordan Brouwer Separation Theorem S^n is separated into two components by H_f. The point set corresponding to (X, W) is a full dimensional open subset of the point set with vector W in $\mathcal{H}_{|A}$, which is in the boundary of V (Lemma 4). In any neighborhood of a point in (X, W) there have to be points of V with coordinates (X, V) with respect to the original system. ■

Recall, that the definition of an antitonism requires the domain of the map to be a lattice where the top element has been removed. Therefore we have to verify that $\mathcal{P} \cup \mathbb{1}_{\mathcal{P}}$ is a lattice. First, we show the existence of non-trivial joins, i. e. those which are not equal to $\mathbb{1}_{\mathcal{P}}$.

Lemma 6. *Assume X, Y are pseudo cells of a sphere system $\mathcal{H}$ such that $sep\,(X, Y) = \emptyset$. Then there is a pseudo cell Z in the sphere system satisfying $Z_{|supp\,(X)} = X_{|supp\,(X)}$, $Z_{|supp\,(Y)} = Y_{|supp\,(Y)}$ and $supp\,(Z) = supp\,(X) \cup supp\,(Y)$.*

Proof. We proceed by induction on $n = |E|$ the case $n = 1$ being trivial. Thus let $n > 1$. If $\text{supp}\,(X) \cup \text{supp}\,(Y) \subset E$ the claim follows immediately from inductive assumption and Lemma 4. We may assume that there is an $f \in E : X_f \neq 0 = Y_f$. In the sphere system $\mathcal{H} \setminus f$ all points of Y belong to the same pseudo cell Y'. By inductive assumtion applied to a $\mathcal{H} \setminus f$ there is a pseudo cell $\tilde{Z}$ which is as desired with respect to $X \setminus f$ and Y'. The claim now again follows from the fact that any point in the pseudo cell X has sufficiently small neighborhood not intersecting H_f. ■

Now we are prepared to prove

Lemma 7. *$\hat{\mathcal{P}}$ is a lattice and a combinatorial manifold.*

Proof. According to Lemma 3 $\mathcal{P}$ is pure dimensional. To verify the remaining conditions we proceed by induction on the dimension d of $\hat{\mathcal{P}}$.

$d = 1$ The only possible partial order is the diamond satisfying all conditions.

$d = 2$ The poset of any such sphere system must be the pseudo cell complex of an even cycle.

$d > 2$ Since $\bigcap_{e \in E} H_e$ separates S^n only if it is a hypersphere, Lemmas 4 and 5 and inductive hypothesis imply that open connectivity and the diamond property are satisfied.

We are left to verify that $\mathcal{P}$ is a lattice. Thus let U be a lower bound for $X_1, X_2 \in \mathcal{P}$. Let $A = \mathrm{cl}\,(\mathrm{sep}\,(X_1, X_2) \cup (\varphi(X_1) \cup \varphi(X_2))$. The subset $M \subseteq S^n$ with signed vector U has to be part of $F = \bigcap_{e \in A} H_e$. Thus Lemma 6 guarantees the existence of a unique maximal lower bound.

Since the meet of two upper bounds always is an upper bound again, we are done.

■

Next we are going to verify the properties of φ.

Lemma 8. *The map $\varphi : \mathcal{P} \to L$ is an antitonic folding.*

Proof. We have already proven that φ is an order reversing, cover preserving surjection from a combinatorial manifold onto a geometric lattice. Consider a pseudo cell X being the join of X_1 and X_2. Lemma 6 guarantees that if $X_1 \vee X_2 \neq \mathbb{1}_{\mathcal{P}}$ we have $\varphi(X_1 \vee X_2) = \varphi(X_1) \wedge \varphi(X_2)$.

The validity of the folding property and local flatness follows from Lemmas 4 and 5 and the observations in the proof of the last lemma. ■

Theorem 4. *The set of signed vectors given by a sphere system is the set of signed covectors of an an oriented matroid.*

Proof. The symmetry of the antitonic folding is immediate from the Jordan Brouwer Separation Theorem since the two pseudo cells of any minimal non-zero signed vector must be separated by each sphere not containing them. ■

3. The crossing property

Considering the results of the last section in more detail we find that axiom ($S2$) splits into two conditions.

(S2a) If $F \not\subseteq H_e^0$ is a flat then $F \cap H_e^0$ is a hypersphere of F.

(S2b) Under the assumptions of (S2a) the components of $F \setminus H_e^0$ are on different sides of H_e^0.

To prove that the map φ is an antitonic folding condition (S2b) (which Edmonds called the crossing property) was not used at all. It was needed only to prove the symmetry.

As it is known that, given a matroid, a cocircuit signature is an oriented matroid cocircuit signature iff the signed cocircuit elimination axiom is satisfied for modular pairs of cocircuits one might ask whether it is possible to restrict condition (S2b) to the cases where F is 1-dimensional. It is immediately clear that this way we always will get a proper oriented matroid cocircuit sign pattern on the 0-dimensional cells. But our above examinations actually yield the following stronger result.

Theorem 5. *Let $\mathcal{H} = (H_e)_{e \in E}$ be a finite family of hyperspheres of S^n. Then $\mathcal{H}$ is a sphere system if and only if*

($S1$) *For all* $A \subseteq E$ $\bigcap_{e\in A} H_e$ *is a topological sphere.*

($S2'a$) *The map* $cl : 2^E \to 2^E$, *given by* $cl(A) = \{e \in E \mid H_e \supseteq \bigcap_{f\in A} H_f\}$ *is a matroid closure and its rank function satisfies:* $r(A) = n - \dim(\bigcap_{e\in A} H_e)$.

($S2'b$): *If* X, Y *are copoints of* $M(E, cl)$ *such that* $X \wedge Y$ *is a coline then the two components of*

$$\bigcap_{e\in X} H_e$$

lie in different components of

$$\bigcap_{e\in X\wedge Y} H_e \setminus \bigcap_{e\in Y} H_e$$

Proof. In Lemma 2 we proved that a sphere system satisfies ($S2'a$). Since ($S2'b$) is a special case of ($S2b$) it is clear that any sphere system satisfies the above axioms.

On the other hand the proofs in the last section showed that a finite family of hyperspheres of S^n satisfying ($S1$) and ($S2'a$) canonically induces an antitonic folding. Assume this folding were not symmetric and there were some copoint $X \in L$ and an atom $e \in L$ such that there exists a path in $\mathcal{P} \setminus 0_{\mathcal{P}}$ between the two preimages of X missing e. Clearly we must have $X \neq 0_L$ and, therefore we may choose some coline $G \in L$ which is covered by X. Now X and $Y := G \vee e$ contradict ($S2'b$). ∎

4. Shellable spheres and lattices

To get a lattice-theoretical characterization of oriented matroids similar to Theorem 2 from the above it is necessary to review some facts from combinatorial topology. The required material is completely covered by the Appendix 4.7 from [1].

The following definition is a lattice-theoretical counterpart of shellability of regular cell complexes.

Definition 6 (cf. [1] 4.7.17). *Let* P *be a finite lattice with rank funktion. A linear ordering* $x_1, \ldots, x_t$ *of its coatoms is a* recursive coatom ordering *if either* $r(1_P) \leq 2$, *or if* $r(1_P) > 2$ *and for all* $1 \leq j \leq t$

1. $[x_j]$, *the ideal generated by* x_j *has a recursive coatom ordering* $y_1, \ldots, y_s$ *and*

2. *there is an* $1 \leq k \leq s$ *such that* $[x_j] \cap \left(\bigcup_{i<j}[x_i]\right) = \bigcup_{i\leq k}[y_i]$.

The notion of recursive coatom ordering is important for us since the big face lattice of an oriented matroid admits a recursive coatom ordering and the following Theorem therefore allows to construct, given an oriented matroid $\mathcal{O}$, a sphere systems which is a topological model for $\mathcal{O}$.

Theorem 6 (cf. [1] 4.7.24). *Let* P *be a finite lattice with rank function* r *and* $r(1_L) = d+2$. *Then* P *is isomorphic to the face poset of some shellable regular cell decomposition* Δ *of the d-sphere if and only if it has the diamond property and admits a recursive coatom ordering.*

Furthermore Δ *is uniquely determined by* P *up to cellular homeomorphism.*

Definition 7. *A* matroidal sphere lattice *is a tripel* $(L, \mathcal{P}, \psi)$, *where* L *is a geometric lattice,* $\mathcal{P}$ *is a poset and* ψ *is a surjective map* $\psi : \mathcal{P} \to L$ *satisfying*

1. $\forall p, q \in \mathcal{P} : (p > q \Longrightarrow \psi(q) > \psi(p))$,

2. $\forall x \in L : \psi^{-1}([x, 1\!\!1_L])$ *with a top element* $1_{\psi^{-1}([x, 1\!\!1_L]}$ *admits a recursive coatom ordering and has the diamond property.*

It is well known that given the big face lattice $\mathcal{P}$ of an oriented matroid, L the geometric lattice of the underlying matroid and φ the zero map then the tripel $(L, \mathcal{P}, \psi)$ satisfies the above conditions. In the following we will prove that vice versa any matroidal sphere lattice arises from an oriented matroid this way.

By the above theorem, given a matroidal sphere lattice $(L, \mathcal{P}, \psi)$ $\mathcal{P}$ has a topological realization $\hat{\mathcal{F}}(\Delta)$ as a regular cell decomposition of the sphere. Let isom denote the isomorphism from Δ to $\mathcal{P}$ and $\varphi = \psi \circ$ isom . By Theorem 6 we have that for all atoms of $e \in L$ the space $\|\varphi^{-1}([e, 1_L])\|$ is a hypersphere H_e of $\|\Delta\|$. Denote by E the set of all atoms of L and let $A \subseteq E$. Then

$$
\begin{aligned}
\bigcap_{e\in E} H_e &= \bigcap_{e\in A} \|\varphi^{-1}([e,1])\| \\
&= \|\text{ isom }^{-1}(\bigcap_{e\in A} \psi^{-1}([e,1]))\| \\
&= \|\text{ isom }^{-1}(\{Z \in \mathcal{P} \mid \forall e \in A : \psi(Z) \geq e\})\| \\
&= \|\text{ isom }^{-1}(\{Z \in \mathcal{P} \mid \psi(Z) \geq \bigvee_{e\in A} e\})\| \\
&= \|\varphi^{-1}(\bigvee_{e\in A} e)\}\|.
\end{aligned}
$$

Therefore, the collection of spheres $(H_e)_{e\in E}$ satisfies $(S1)$. A similar computation verifies that $(S2'a)$ holds as well.

Obviously $(S2'b)$ will not be valid without further assumptions. This is easily seen on a rank 2 example. But $(S2'b)$ is easily translated into our lattice theoretical notation.

Definition 8. *A matroidal sphere lattice* $(L, \mathcal{P}, \psi)$ *is* symmetric, *if for all copoints* $X \in L$ *and all colines* $G \in L$ *such that* X *covers* G *we have* $\psi^{-1}([G, 1_L]) \setminus \psi^{-1}([X, 1_L])$ *consists of two connected components of the same cardinality.*

Theorem 7. *Let* $(L, \mathcal{P}, \varphi)$ *be a symmetric matroidal sphere lattice. Then* $\mathcal{P}$ *is the partial order of an oriented matroid and* φ *is its zero map.*

Proof. By the above remarks we are left to verify that $(S2'b)$ holds. Assume to the contrary that X, Y are copoints of $M(E, \mathrm{cl})$ such that $X \wedge Y$ is a coline and the two components of

$$\bigcap_{e\in X} H_e$$

were in the same component of

$$\bigcap_{e\in X\wedge Y} H_e \setminus \bigcap_{e\in Y} H_e.$$

Consider the two components A_1, A_2 of $\psi^{-1}([X \wedge Y, 1_L]) \setminus \psi^{-1}([X, 1_L])$. By symmetry they have the same cardinality. Now let B_1 and B_2 denote the two components of $\psi^{-1}([X \wedge Y, 1_L]) \setminus \psi^{-1}([Y, 1_L])$. By assumption wlog. $A_1 \subset B_1$ and thus $B_2 \subset A_2$ and we get

$$|A_1| < |B_1| = |B_2| < |A_2|$$

contradicting symmetry. ■

References

[1] Björner A., Las Vergnas M., Sturmfels B., White N., Ziegler G. Oriented Matroids. Cambridge: Cambridge University Press, 1993.

[2] Bland R.G., Las Vergnas M. Orientability of Matroids // J. of Combinatorial Theory, Series B, Vol. 24, 1978, P. 94-123.

[3] Bokowski J., Mock S., Streinu I. The Folkman-Lawrence Topological Representation Theorem for Oriented Matroids – An Elementary Proof in Rank 3 // European J. of Combinatorics (to appear).

[4] Cordovil R., Fukuda K. Oriented Matroids and Combinatorial Manifolds // European J. of Combinatorics, Vol. 14, 1993, P. 9-15.

[5] Hochstättler W. Seitenflächenverbände orientierter Matroide. Dissertation, Universität zu Köln, 1992.

[6] Hochstättler W. A lattice-theoretical characterization of oriented matroids // European J. of Combinatorics, Vol. 18, 1997, P. 563-574.

[7] Hocking J.G., Young G.S. Topology. London: Addison Wesley, 1961.

[8] Folkman J., Lawrence J. Oriented Matroids // J. of Combinatorial Theory, Series B, Vol. 25, 1978, 1P. 99-236.

[9] Kinsey L.C. Topology of Surfaces. N. Y.: Springer, 1993.

[10] Mandel A. Topology of Oriented Matroids. Ph.D. Thesis. University of Waterloo, 1982.

[11] Rushing T. B. Topological Embeddings. N. Y., London: Academic Press, 1973.

[12] White N.(ed.) Theory of Matroids. Cambridge: Cambridge University Press, 1986.

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